Addition invariance on bi-partitions, nullified and positive players, and weighted division values

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Abstract

New axioms for cooperative games with transferable utilities are introduced. The nullified player axiom compares a game before and after a specified player becomes null, *i.e.*, the worth of a coalition in this new game is the worth of the coalition without the specified player in the original game. The axiom requires that if the player who becomes null loses from such a change, then every other player should lose too. The positive player axiom requires to assign a positive payoff to a player that belongs to coalitions with positive worth only. The axiom of addition invariance on bi-partitions requires that the payoff vector recommended by a value should not be affected by an identical change in worth of both a coalition and the complementary coalition. We study the consequence of imposing some of these axioms in addition to some classical axioms. It turns out that the resulting values or set of values have all in common to split efficiently the worth achieved by the grand coalition according to an exogenously given weight vector. As a result, we also obtain characterizations of the equal division value.

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1. Introduction

The axioms employed to design values in cooperative game theory with transferable utilities can be divided up into punctual and relational axioms. A punctual axiom applies to each game separately and a relational axiom relates payoff vectors of games that are related

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in a certain way. This article introduces one new punctual axiom and two new relational axioms.

The well-established null player axiom and nullifying player axiom are punctual axioms. The former axiom recommends to assign a zero payoff to a null player, *i.e.*, a player with zero contributions to coalitions. The latter axiom enforces a zero payoff to a nullifying player, *i.e.*, a player belonging to coalitions with zero worth only. These two axioms play important roles since they enable to distinguish the Shapley value (Shapley, 1953) from the equal division value, two values built on opposite equity principles (see van den Brink, 2007).

There exist few more axioms that rest on the null and nullifying players, or on variants of these types of players. Two examples are the null player out axiom (Derks and Haller, 1999) and the null player in a productive environment axiom (Casajus and Huettner, 2013). The former is a relational axiom stating that removing a null player from a game does not affect the payoff of the remaining players. The latter is a punctual axiom that specifies to assign a non-negative payoff to a null player if the grand coalition has a non-negative worth.

In this article, we call upon two variants of the null player and nullifying player axioms. The first one is the nullified player axiom, which compares a game before and after a specified player becomes null, *i.e.*, the worth of a coalition in this new game is the worth of the coalition without the specified player in the original game. The nullified player axiom is a relational axiom requiring that if the specified player loses from such a change, then every other player should lose too, albeit the magnitudes of these losses can vary. As such, this axiom possesses some flavor of the solidarity principle as defined by Thomson (2012).The second one is the positive player axiom, a punctual axiom that requires to assign a positive payoff to a positive player (a player belonging to coalitions with positive worth only). Any nullifying player is also a positive player, although the nullifying player and positive player axioms are not related to each other, *i.e.*, neither axiom implies the other.

The vast category of relational axioms includes as a subclass the axioms of invariance. Such axioms specify either the same payoff vector or the same payoff for some specific players across games that are related in certain ways. Besides the null player out axiom, a well-known example of axiom of invariance is the axiom of marginality (Young, 1985), which requires to attribute the same payoff to a player in two games where his contributions to coalitions are identical. Further axioms of invariance are discussed by Béal et al. (2012).

We introduce a new axiom of invariance relying on the idea of bi-partitions, which dates back to von Neumann and Morgenstern (1944). More specifically, our axiom of addition invariance on bi-partitions states that the chosen payoff vector should not be affected by an identical change in worth of both a coalition and the complementary coalition. We show that this axiom is equivalent to self-duality if one restricts to the domain of additive values.

We study the consequence of imposing some of these axioms in addition to some classical axioms such as efficiency, additivity, linearity, or the equal treatment axiom. It turns out that the resulting values or classes of values have all in common to split efficiently the worth achieved by the grand coalition according to an exogenously given weight vector summing up to unity. We refer to the weighted division values when the weight vector can contain negative coordinates, and to the positively weighted division values for the subclass of weighted division values with non-negative weights. Naturally, the equal division value is the unique weighted division value with identical weights. All in all, the article contains ten characterizations of such values or classes of values. To the best of our knowledge, the only similar article in cooperative game theory is due to van den Brink (2009) who obtains a characterization of the class of all weighted division values by imposing the axiom of collusion neutrality (see Haller, 1994) in addition to linearity and efficiency.

The weighted division values constitute an interesting class of values for at least two reasons. Firstly, although the requirement to treat substitute players equally appears to be natural in many situations, it is desirable to have the option of treating substitute players differently in order to reflect exogenous characteristics, such as income or health status. This can be achieved by incorporating exogenous weights into the construction of a value. Weighted values have been popularized by Kalai and Samet (1987) who study the weighted Shapley values. In a sense, the weighted division values generalize the equal division value as the weighted Shapley values generalize the Shapley value. Secondly, proportional division methods are very often employed in a lot of applications such as claim problems, cost allocation problems, insurance, law and so on. We refer to Tijs and Driessen (1986), Lemaire (1991), Balinski and Young (2001), and Thomson (2003) for rich surveys, and to Chun (1988), Moulin (1987), and Thomson (2013) for proportional division methods that rest on exogenously given weights.

Our study possesses other advantages. From a theoretical point of view, the axiomatic characterizations of the equal division value always rest on at least one of the classical axioms of efficiency, the equal treatment axiom, or linearity/additivity. Some of our results avoid to use some of or all these axioms. As an example, we prove that the equal division value is characterized by addition invariance on bi-partitions, the nullifying player axiom, and weak covariance, where this last axiom is a weak version of covariance in the sense that the added additive game is symmetric. Moreover, two of our characterizations of the positively weighted division values give insight into the role of the equal treatment axiom in the characterizations of the equal division value. While the role of the equal treatment axiom is obvious in these two characterizations of the equal division value, it is more difficult to grasp in the characterization provided by van den Brink (2007).

The rest of the article is organized as follows. Section 2 presents the basic material about cooperative games with transferable utilities. Section 3 introduces the axiom of addition invariance on bi-partitions, and contains all the results in which this axiom is invoked. Section 4 introduces the positive player and nullified player axioms, and offers the results mobilizing these axioms. A comparison with the main result in van den Brink (2007) is provided in Section 5. Section 6 concludes. Finally, the logical independence of the axioms used in each of our characterizations is demonstrated in the appendix.

2. Basic definitions and notations

Let $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$, be the set of players, which is fixed throughout the article. A TU-game on N, or simply a game, is given by the **coalition function** $v \in \mathbb{V} := \{f : 2^N \longrightarrow \mathbb{R} \mid f(\emptyset) = 0\}$. Subsets of N are called coalitions. We write i instead of $\{i\}$ for each singleton coalition. The size of a coalition S is denoted by its lower-case version s; and $v(S)$ is called the worth of coalition S.

For all $c \in \mathbb{R}$, the **symmetric additive game** induced by c is denoted by $\mathbf{c}$ and is given by $\mathbf{c}(S) = s \cdot c$ for all $S \subseteq N$. The particular case $c = 0$ gives rise to the **null game 0** given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For $v, w \in \mathbb{V}$ and $c \in \mathbb{R}$, the coalition functions $v + w$ and $c \cdot v$ are given by $(v + w)(S) = v(S) + w(S)$ and $(c \cdot v)(S) = c \cdot v(S)$ for all $S \subseteq N$. For $\emptyset \subsetneq T \subseteq N$, the game e_T given by $e_T(S) = 1$ if $S = T$ and $e_T(S) = 0$ for $S \neq T$ is called the **standard game** induced by T. Obviously, any $v \in \mathbb{V}$ admits a unique representation in terms of standard games:

$$v = \sum_{\emptyset \subsetneq T \subseteq N} v(T) \cdot e_T. \tag{1}$$

For $\emptyset \subsetneq T \subseteq N$, the game u_T given by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ if $S \not\supseteq T$ is called the **unanimity game** induced by T. The **dual** of a game v is the game v^D given by $v^D(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

Player $i \in N$ is **null** in $v \in \mathbb{V}$ if $v(S) = v(S \setminus i)$ for all $S \ni i$. Player $i \in N$ is **nullifying** in $v \in \mathbb{V}$ if $v(S) = 0$ for all $S \ni i$. Player $i \in N$ is **positive** in $v \in \mathbb{V}$ if $v(S) \geq 0$ for all $S \ni i$. Two players $i, j \in N$ are **substitutes** in $v \in \mathbb{V}$ if $v(S \cup i) = v(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$.

A **value** is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^n$ to any $v \in \mathbb{V}$. We consider the following values. Let $\Delta^n := \left\{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = 1\right\}$ and $\Delta^n_+ := \Delta^n \cap \mathbb{R}^n_+$. For $\omega \in \Delta^n$, the **ω-weighted division value** WD^ω is given by

$$\mathrm{WD}^\omega_i(v) = \omega_i \cdot v(N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The class of all weighted division values is denoted by $\mathcal{W}$,

$$\mathcal{W} = \{\varphi \mid \text{there is } \omega \in \Delta^n \text{ s.t. } \varphi = \mathrm{WD}^\omega\};$$

the class of positively weighted division values $\mathcal{W}_+ \subseteq \mathcal{W}$ is given by

$$\mathcal{W}^+ = \left\{\varphi \mid \text{there is } \omega \in \Delta^n_+ \text{ s.t. } \varphi = \mathrm{WD}^\omega\right\}.$$

Note that the constants ω_i, $i \in N$, in the definitions of the weighted division values are exogenously given, *i.e.*, they do not depend on the game v under consideration. The **equal division value (ED-value)** is the positively weighted division value given by

$$\mathrm{ED}_i(v) = \frac{v(N)}{n} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Shapley value** (Shapley, 1953) is given by:

$$\mathrm{Sh}_i(v) = \sum_{S \subseteq N : S \ni i} \frac{(n-s)! \cdot (s-1)!}{n!} \cdot (v(S) - v(S \setminus i)) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Later on, we will use the following standard axioms for values.

Efficiency, E. For all $v \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Equal treatment axiom, ET. For all $v \in \mathbb{V}$ and $i, j \subseteq N$ such that i and j are substitutes in v, $\varphi_i(v) = \varphi_j(v)$.

Null player axiom, N. For all $v \in \mathbb{V}$ and $i \in \mathbb{V}$ such that i is null in v, $\varphi_i(v) = 0$.

Nullifying player axiom, Nf. For all $v \in \mathbb{V}$ and $i \in N$ such that i is nullifying in v, $\varphi_i(v) = 0$.

Additivity, A. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Linearity, L. For all $v, w \in \mathbb{V}$ and all $c \in \mathbb{R}$, $\varphi(c \cdot v + w) = c \cdot \varphi(v) + \varphi(w)$.

Self-duality, SD. For all $v \in \mathbb{V}$, $\varphi(v) = \varphi(v^D)$.

The Shapley value can be characterized by efficiency, additivity, the null player axiom and the equal treatment axiom. Replacing the null player axiom by the nullifying player axiom yields a characterization of the ED-value.

Proposition 1 (van den Brink, 2007). *The ED-value is the unique value that satisfies additivity (**A**), efficiency (**E**), the equal treatment axiom (**ET**), and the nullifying player axiom (**Nf**).*

3. Addition invariance on bi-partitions

The use of bi-partitions of N has been suggested by von Neumann and Morgenstern (1944). Suppose that in a game $v \in \mathbb{V}$ the grand coalition N splits into two coalitions S and $N \setminus S$ that bargain on the surplus $v(N) - v(S) - v(N \setminus S)$ they can create by cooperating. In a sense, the worths $v(S)$ and $v(N \setminus S)$ are the bargaining powers of these two bargaining coalitions. The axiom of addition invariance on bi-partitions indicates that if the worths of S and $N \setminus S$ vary by the same amount, then this change should not affect the resulting payoff vector. For $v \in \mathbb{V}$, $\emptyset \subsetneq S \subsetneq N$, and $c \in \mathbb{R}$, the game $v_{S,c} \in \mathbb{V}$ induced by v, S and c is given by

$$v_{S,c}(T) := \begin{cases} v(T) + c, & T \in \{S, N \setminus S\}, \\ v(T), & T \in 2^N \setminus \{S, N \setminus S\} \end{cases} \qquad \text{for all } T \subseteq N. \tag{2}$$

Addition invariance on bi-partitions, AIB. For all $v \in \mathbb{V}$, $\emptyset \subsetneq S \subsetneq N$, and $c \in \mathbb{R}$, $\varphi(v) = \varphi(v_{S,c})$.

The next proposition highlights that addition invariance on bi-partitions is equivalent to self-duality for additive values.

Proposition 2. *(a) If the value φ satisfies addition invariance on bi-partitions (**AIB**), then φ satisfies self-duality (**SD**).*

*(b) If the value φ satisfies self-duality (**SD**) and additivity (**A**), then φ satisfies addition invariance on bi-partitions (**AIB**).*

Proof. *(a)*: Pick any value φ that satisfies **AIP**. Consider any game $v \in \mathbb{V}$ and its dual v^D, and define the game $w \in \mathbb{V}$ by $w = (v + v^D)/2$. Now consider any $i \in N$ and any ordering

$(S^1, \ldots, S^{2^{n-1}-1})$ of all coalitions containing player i except N. For all $p \in \{1, \ldots, 2^{n-1}-1\}$, recursively construct the game v^p by $v^p = (v^{p-1})_{S^p,c^p}$, where $v^0 = v$ and

$$c^p = \frac{v^{p-1}(N) - v^{p-1}(S^p) - v^{p-1}(N \setminus S^p)}{2}.$$

At each step p, we have $v^p(T) = v^{p-1}(T)$ for $T \neq S^p$ or $T \neq N \setminus S^p$,

$$v^p(S^p) = \frac{v(S^p) + v^D(S^p)}{2}, \quad \text{and} \quad v^p(N \setminus S^p) = \frac{v(N \setminus S^p) + v^D(N \setminus S^p)}{2}.$$

As a consequence, we obtain $v^{2^{n-1}-1} = w$. Successive applications of **AIB** yield $\varphi(v) = \varphi(w)$. Considering v^D instead of v, *i.e.*, $v^0 = v^D$, and proceeding in the same fashion, we get $\varphi(v^D) = \varphi(w)$. Therefore, $\varphi(v) = \varphi(v^D)$, as desired.

(b): Pick any value φ that satisfies **SD** and **A**. Consider $v \in \mathbb{V}$, $\emptyset \subsetneq S \subsetneq N$, and $c \in \mathbb{R}$, and the game $v_{S,c}$ induced by v, S and c. Then, $v - v_{S,c} = c \cdot (e_S + e_{N\setminus S})$. In addition, for all $T \subseteq N$, we have $c \cdot e_S^D(T) = -c$ if $T = N \setminus S$ and $c \cdot e_S^D(T) = 0$ if $T \neq N \setminus S$. Therefore, $c \cdot e_{N\setminus S} = -(c \cdot e_S)^D = -c \cdot e_S^D$, and we get $(v - v_{S,c}) = c \cdot (e_S - e_S^D)$. By **A** and **SD**, we obtain for all $i \in N$, $0 = \varphi_i(c \cdot (e_S - e_S^D))$ and so $0 = \varphi_i(c \cdot (e_S - e_S^D)) = \varphi_i(v - v_{S,c})$. Applying **A** once more, we obtain $\varphi_i(v) = \varphi_i(v_{S,c})$ for all $i \in N$, as desired. □

Proposition 2 (b) implies that the Shapley value as well as any weighted division value satisfy addition invariance on bi-partitions.

Remark 3. To see why the converse of Proposition 2 (a) fails, consider the non-additive value φ given by

$$\varphi_i(v) = (v(N) - v(N \setminus i) - v(i))^2 \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

This value satisfies self-duality but not addition invariance on bi-partitions.

We provide a characterization of the class of weighted division values obtained by dropping the equal treatment axiom and additivity from Proposition 1, and adding addition invariance on bi-partitions and linearity.

Proposition 4. *A value φ satisfies efficiency (**E**), linearity (**L**), addition invariance on bi-partitions (**AIB**), and the nullifying player axiom (**Nf**) if and only if $\varphi \in \mathcal{W}$.*

Proof. It is clear that all values $\varphi \in \mathcal{W}$ satisfies **L**, **E**, **AIB**, and **Nf**. Reciprocally, let the value φ satisfy **L**, **E**, **AIB**, and **Nf**. Consider any non-empty coalition $S \subsetneq N$ and any real number $c \in \mathbb{R}$. Note that $v_{S,c} = v + c \cdot (e_S + e_{N\setminus S})$. By **L** and **AIB**, $\varphi(e_S) = -\varphi(e_{N\setminus S})$. Next, fix any $i \in N$ and any coalition $S \ni i$, $S \neq N$. Since i is nullifying in $e_{N\setminus S}$, we get $\varphi_i(e_{N\setminus S}) = 0$ by **Nf**. By **AIB**, $\varphi_i(e_S) = -\varphi_i(e_{N\setminus S}) = 0$. Thus, $\varphi_i(e_S) = 0$ for all $S \neq N$ and all $i \in N$. Since $\{S, N \setminus S\}_{S \ni i, S \neq N} = \{S\}_{S \subsetneq N}$, **L** and (1) imply

$$\varphi_i(v) = v(N) \cdot \varphi_i(e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Set $\omega_i = \varphi_i(e_N)$, $i \in N$, and conclude by **E** that $\omega \in \Delta^N$ and $\varphi = \mathrm{WD}^\omega$, *i.e.*, $\varphi \in \mathcal{W}$. $\square$

The necessity to strengthen additivity used in Proposition 1 by invoking linearity in Proposition 4 and other results is explained in conclusion of the article. From Propositions 2 and 4, we get the following corollary.

Corollary 5. *A value φ satisfies efficiency (**E**), linearity (**L**), self-duality (**SD**), and the nullifying player axiom (**Nf**) if and only if $\varphi \in \mathcal{W}$.*

Most of the characterizations of the ED-value in the literature use efficiency, the equal treatment axiom, or additivity. The following axiom enables a characterization of the ED-value without any of these axioms.

Weak covariance, Co$^-$. For all $v \in \mathbb{V}$ and $i \in N$, and all $c, c' \in \mathbb{R}$, $\varphi_i(c'\cdot v+\mathbf{c}) = c'\cdot\varphi_i(v)+c$.

Weak covariance is a weaker version of the classical axiom of covariance since in the latter the added additive game needs not to be symmetric. Weak covariance is also imposed by Béal et al. (2012, 2013) and van den Brink et al. (2012).

Remark 6. Note that any value φ satisfying linearity (**L**), additivity (**A**), or weak covariance (**Co$^-$**) is an odd function, *i.e.*, $\varphi(-v) = -\varphi(v)$ for all $v \in \mathbb{V}$.

We show that replacing linearity and efficiency in Proposition 4 by weak covariance singles out the ED-value within the class of all weighted values.

Proposition 7. *The ED-value is the unique value that satisfies addition invariance on bi-partitions (**AIB**), the nullifying player axiom (**Nf**), and weak covariance (**Co$^-$**).*

Proof. One easily checks that the ED-value satisfies **AIB**, **Nf**, and **Co$^-$**. To prove the uniqueness part, consider any value φ that satisfies **AIB**, **Nf**, and **Co$^-$**. Pick any game v and define the game $v^0 := v + (-v(N)/n) \cdot \sum_{j\in N} u_j$. Note that v^0 is the sum of v and the symmetric additive game induced by $(-v(N)/n)$, and that $v^0(N) = 0$. Now fix a player $i \in N$, and consider any ordering $(S^1, \ldots, S^{2^{n-1}-1})$ of all coalitions containing i except N. For all $p \in \{1, \ldots, 2^{n-1}-1\}$ construct recursively the game v^p as $v^p = (v^{p-1})_{S^p,-v(S^p)}$. As a result, the game $v^{2^{n-1}-1}$ is such that $v^{2^{n-1}-1}(S) = 0$ for all coalitions S containing player i. This means that i is a nullifying player in this game and so $\varphi_i(v^{2^{n-1}-1}) = 0$ by **Nf**. By successive applications of **AIB** and **Co$^-$**, we get

$$0 = \varphi_i(v^{2^{n-1}-1}) = \varphi_i(v^0) = \varphi_i(v) - \frac{v(N)}{n} \qquad \text{for all } i \in N,$$

i.e., $\varphi_i(v) = \mathrm{ED}_i(v)$. Because v and i were chosen arbitrarily the proof is complete. $\square$

From Propositions 2 (b) and 7, we obtain the following corollary, for which the logical independence of the axioms is preserved as shown in the appendix.

Corollary 8. *The ED-value is the unique value that satisfies additivity (**A**), self-duality (**SD**), the nullifying player axiom (**Nf**), and weak covariance (**Co$^-$**).*

4. Null, nullified, and positive players

This section invokes three extra axioms, which rest on the notions of the null player, the nullifying player, and on a variant of these types of players. The first of these axioms is introduced by Casajus and Huettner (2013) and requires that if the grand coalition enjoys a non-negative worth, then a null player should not be attributed a negative payoff.

Null player in a productive environment axiom, NPE. For all $v \in \mathbb{V}$ and $i \in N$ such that $v(N) \geq 0$ and i is a null player in v, $\varphi_i(v) \geq 0$.

Casajus and Huettner (2013) employ the null player in a productive environment axiom in order to characterize mixtures between the Shapley value and the ED-value. Dropping addition invariance on bi-partitions from Proposition 4 and adding the null player in a productive environment axiom selects the positively weighted division values among the set of all weighted division values.

Proposition 9. *A value φ satisfies efficiency (**E**), linearity (**L**), the null player in a productive environment axiom (**NPE**), and the nullifying player axiom (**Nf**) if and only if $\varphi \in \mathcal{W}^+$.*

Proof. It is clear that all values $\varphi \in \mathcal{W}^+$ satisfies **E**, **L**, **NPE**, and **Nf**. Reciprocally, let the value φ satisfy **E**, **L**, the **NPE**, and **Nf**. We first show that $\varphi_i(e_S) = 0$ for all $S \ni i$, $S \neq N$. Pick any player $i \in N$ and any coalition $S \ni i$ with $1 < s < n$. Consider the game w_S^i given by $w_S^i = e_S + e_{S \setminus i}$. Player i is null in w_S^i and $w_S^i(N) = 0$ since $s < n$. By **L** and **NPE**, we get $\varphi_i(e_S) \geq -\varphi_i(e_{S\setminus i})$. Taking $-w_S^i$ instead of w_S^i, we have $-\varphi_i(e_S) \geq \varphi_i(e_{S\setminus i})$. Thus, $\varphi_i(e_S) = -\varphi_i(e_{S\setminus i})$. In each standard game $e_{S\setminus i}$, player $i \in S \subseteq N$ is nullifying, so that **Nf** yields $\varphi_i(e_{S\setminus i}) = 0$. It follows that $\varphi_i(e_S) = -\varphi_i(e_{S\setminus i}) = 0$ for all $S \ni i$ such that $1 < s < n$. Moreover, since all $j \in N \setminus i$ are nullifying in e_i, we also get $\varphi_j(e_i) = 0$. Thus, **E** in e_i implies that $\varphi_i(e_i) = 0$ as well. Furthermore, since i is nullifying in $e_{N\setminus i}$, it holds that $\varphi_i(e_{N\setminus i}) = 0$. As a consequence, $\varphi_i(e_S) = 0$ for all $S \neq N$ and all $i \in N$ as claimed, which implies that (1) can be rewritten as

$$\varphi_i(v) = v(N) \cdot \varphi_i(e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

By **NPE**, we obtain $\varphi_i(e_N) \geq -\varphi_i(e_{N\setminus i}) = 0$. Set $\omega_i = \varphi_i(e_N) \geq 0$ for all $i \in N$ and conclude by efficiency that $\varphi \in \mathcal{W}^+$. □

The second axiom defined in this section is new. It stipulates that if the worths of all coalitions to which a given player belongs are non-negative, then this player should not be attributed a negative payoff.

Positive player axiom, PP. For all $v \in \mathbb{V}$ and $i \in N$ such that i is a positive player in v, $\varphi_i(v) \geq 0$.

The positive player axiom is related to the nullifying player axiom in a similar way as the null player axiom is related to the null player in a productive environment axiom. Replacing the null player in a productive environment axiom and the nullifying player axiom in Proposition 9 by the positive player axiom gives and alternative characterization of the positively weighted division values.

Proposition 10. *A value φ satisfies efficiency (**E**), linearity (**L**), and the positive player axiom (**PP**) if and only if $\varphi \in \mathcal{W}^+$.*

Proof. In a game $v \in \mathbb{V}$, a player $i \in N$ is positive only if $v(N) \geq 0$. Thus, any value $\varphi \in \mathcal{W}^+$ satisfies the **PP**. For the uniqueness part, consider any value satisfying the three axioms. For $S \neq N$, all players are positive in e_S. By **PP**, $\varphi_i(e_S) \geq 0$ for all $i \in N$. By **E**, it must be that $\varphi_i(e_S) = 0$ for all $i \in N$. In e_N, the **PP** also implies $\varphi_i(e_N) \geq 0$ for all $i \in N$. Set $\omega_i = \varphi_i(e_N) \geq 0$ for all $i \in N$. Conclude by **E**, **L**, and (1) that $\varphi \in \mathcal{W}^+$. $\square$

The third axiom that we define in this section is also new and incorporates a solidarity principle, which is described informally by Thomson (2012) as follows:

> "When the circumstances in which some group of agents find themselves change — the group could be the entire population of agents present or some subset — and if none of them bears any particular responsibility for the change, or deserves any particular credit for it, their welfare should be affected in the same direction: all members of the group should end up at least as well off as they were initially, or they should all end up at most as well as off as they were initially."

The nullified player axiom compares a game before and after a specified player becomes null in the sense that he now has a null contribution to all coalition he belongs to. The axiom simply requires uniformity in the direction of the payoff variation for all players in the situations where the considered player loses from being nullified. As such, the nullified player axiom is silent, a priori, on what happens if this player increases his payoff after being nullified. Formally, for a game $v \in \mathbb{V}$ and a player $i \in N$, the associated game in which i is nullified, denoted by $v^{\mathbf{N}i} \in \mathbb{V}$, is given by

$$v^{\mathbf{N}i}(S) = v(S \setminus i) \qquad \text{for all } S \subseteq N. \tag{3}$$

Nullified player axiom, Nd. For all $v \in \mathbb{V}$ and $i, j \in N$, $\varphi_i(v) \geq \varphi_i(v^{\mathbf{N}i})$ implies $\varphi_j(v) \geq \varphi_j(v^{\mathbf{N}i})$.

The nullified player axiom can be related to the axiom of population solidarity proposed by Chun and Park (2012), which requires that if some players leave a game, then the remaining players should be affected in the same direction. When a player is nullified, he does not exactly leave the game, but his presence or absence in a coalition has no impact of the achieved worths. In Chun and Park (2012), population solidarity belongs to the set of axioms characterizing the equal surplus division value on the class of games with variable player sets.

Remark 11. If a player $j \in N$ is null in a game v, then he remains null after another player $i \in N \setminus j$ is nullified, *i.e.* j remains null in game $v^{\mathbf{N}i}$.

The next result shows that the nullified player axiom can be used as a substitute to the positive player axiom in Proposition 10 in order to provide another characterization of the positively weighted division values.

Proposition 12. *The value φ satisfies efficiency (**E**), linearity (**L**), and the nullified player axiom (**Nd**) if and only if $\varphi \in \mathcal{W}^+$.*

Proof. Any value $\varphi \in \mathcal{W}^+$ satisfies the three axioms. For the uniqueness part, consider any value φ that satisfies the three axioms. For all $S \subseteq N$ and $i \in S$, $(e_S)^{\mathbf{N}i} = \mathbf{0}$. By **L**, $\varphi_j((e_S)^{\mathbf{N}i}) = \varphi_j(\mathbf{0}) = 0$ for all $j \in N$. Next, we show that $\varphi_i(e_S) \geq 0$ for all $S \subseteq N$ and all $i \in S$. By contradiction, assume that $\varphi_i(e_S) < 0$. Consider the game $-e_S$. By **L**, we get $\varphi_i(-e_S) = -\varphi_i(e_S) > 0$, and of course $(-e_S)^{\mathbf{N}i} = (e_S)^{\mathbf{N}i} = \mathbf{0}$. Thus, $\varphi_i(-e_S) > \varphi_i((-e_S)^{\mathbf{N}i})$. By **Nd**, this implies that $\varphi_j(-e_S) \geq 0$ for all $j \in N \setminus i$. Summing on all $j \in N$, we obtain $\sum_{j\in N} \varphi_j(-e_S) > 0$, or equivalently $\sum_{j\in N} \varphi_j(e_S) < 0$, a contradiction with the fact that φ satisfies **E**. In other words, the inequality $\varphi_i(e_S) \geq 0$ for all $S \subseteq N$ and $i \in S$ is true. Then $\varphi_i(e_S) \geq \varphi_i((e_S)^{\mathbf{N}i}) = 0$ implies $\varphi_j(e_S) \geq \varphi_j((e_S)^{\mathbf{N}i}) = 0$ for all $j \in N$ by **Nd**. It remains to distinguish two cases. Firstly, suppose that $S \neq N$. By **E** and $e_S(N) = 0$, we get $\varphi_j(e_S) = 0$ for all $j \in N$. Secondly, suppose that $S = N$. Set $\omega_j = \varphi_j(e_N) \geq 0$ for all $j \in N$. By **E**, $\sum_{j\in N} \omega_j = 1$. The proof is complete by **L** and (1). $\square$

Replacing linearity in Proposition 12 by weak covariance singles out the ED-value from the set of positively weighted division values. This result shows that the symmetric treatment imposed by weak covariance in the added additive game clearly plays a decisive role, even if the linearity is lost.

Proposition 13. *The ED-value is the unique value that satisfies efficiency (**E**), weak covariance (**Co**$^-$), and the nullified player axiom (**Nd**).*

The proof makes use of the following lemma, which describes two invariance properties across games related by a nullified player.

Lemma 14. *Consider a value φ satisfying efficiency (**E**), the nullified player axiom (**Nd**), and $\varphi(v) = -\varphi(v)$ for all $v \in \mathbb{V}$. Pick any $v \in \mathbb{V}$ and $i \in N$ such that $v(N \setminus i) = v(N)$. Then,*

(a) $\varphi(v) = \varphi(v^{\mathbf{N}i})$,

(b) $\varphi(v) = \varphi(w)$ for all $w \in \mathbb{V}$ such that $w(N) = v(N)$ and $w^{\mathbf{N}i} = v^{\mathbf{N}i}$.

Proof. *(a)* Let $i \in N$ and $v \in \mathbb{V}$ be such that $v(N \setminus i) = v(N)$. By way of contradiction, suppose that $\varphi_i(v) > \varphi_i(v^{\mathbf{N}i})$. By **Nd**, we have $\varphi_j(v) \geq \varphi_j(v^{\mathbf{N}i})$ for all $j \in N \setminus i$. Hence, applying **E** in v and $v^{\mathbf{N}i}$ yields

$$v(N) = \sum_{j\in N} \varphi_j(v) > \sum_{j\in N} \varphi_j(v^{\mathbf{N}i}) = v^{\mathbf{N}i}(N) = v(N),$$

a contradiction. Thus,

$$\varphi_i(v) \leq \varphi_i(v^{\mathbf{N}i}). \tag{4}$$

Suppose now that $\varphi_i(v) < \varphi_i(v^{\mathbf{N}i})$. Since $(-v)^{\mathbf{N}i} = -(v^{\mathbf{N}i})$ and $\varphi_i(-v) = -\varphi_i(v)$, we then get $-\varphi_i(-v) = \varphi_i(v) < \varphi_i(v^{\mathbf{N}i}) = -\varphi_i((-v)^{\mathbf{N}i})$. Therefore, $\varphi_i(-v) > \varphi_i((-v)^{\mathbf{N}i})$, which is a contradiction to (4) for $-v$ instead of v.

We obtain $\varphi_i(v) = \varphi_i(v^{\mathbf{N}i})$. Then, **E** and **Nd** entail $\varphi_j(v) = \varphi_j(v^{\mathbf{N}i})$ for all $j \in N \setminus i$.

(b): Since $w(N) = v(N), w(N) = w(N \setminus i)$, and $w^{\mathbf{N}i} = v^{\mathbf{N}i}$, we have $\varphi(w) = \varphi(w^{\mathbf{N}i})$ by (a) and $\varphi(w^{\mathbf{N}i}) = \varphi(v^{\mathbf{N}i})$. Hence,

$$\varphi(w) = \varphi(w^{\mathbf{N}i}) = \varphi(v^{\mathbf{N}i}) \overset{\text{(a)}}{=} \varphi(v),$$

as desired. □

Using Lemma 14, Proposition 13 can be demonstrated.

Proof. (Proposition 13) Since the ED-value satisfies all the axioms, we shall only prove the uniqueness part. Consider any φ that satisfies the three axioms, which implies that φ is an odd function, and in turn that φ meets the conditions of Lemma 14. Pick any $v \in \mathbb{V}$. To show $\varphi(v) = \mathrm{ED}(v)$. From v, construct the game $r^1 \in \mathbb{V}$ such that $r^1 = v + [v(N \setminus 1) - v(N)] \cdot \sum_{k \in N} u_k$.[1] By $\mathbf{Co}^-$,

$$\varphi_j(r^1) = \varphi_j(v) + v(N \setminus 1) - v(N) \text{ for all } j \in N. \tag{5}$$

It is also easy to check that $r^1(N) = r^1(N \setminus 1)$. Next, from r^1, construct the game $v^1 \in \mathbb{V}$ such that

$$v^1 = \sum_{S \not\ni 1} r^1(S) \cdot e_S + r^1(N) \cdot e_N + \sum_{S \ni 1, S \neq N} s \cdot \frac{r^1(N)}{n} \cdot e_S.$$

Observe that $v^1(N) = r^1(N)$ and $(v^1)^{\mathbf{N}1} = (r^1)^{\mathbf{N}1}$, and recall that $r^1(N) = r^1(N \setminus 1)$. Thus, Lemma 14 (b) yields

$$\varphi(v^1) = \varphi(r^1). \tag{6}$$

Now, from v^1, construct the game $w^1 \in \mathbb{V}$ such that

$$w^1 = v^1 - \frac{v^1(N)}{n} \cdot \sum_{k \in N} u_k.$$

It is easy to verify that for all $S \ni 1$, it holds that $w^1(S) = 0$, *i.e.*, player 1 is nullifying in w^1. Moreover, $\mathbf{Co}^-$ implies that, for all $j \in N$, we have

$$\varphi_j(w^1) = \varphi_j(v^1) - \frac{v^1(N)}{n}. \tag{7}$$

[1] The proof obviously works for any ordering of the players other that the usual ordering $1, \ldots, n$, which we use here and below for the sake of simplicity.

Now, for all $k \in \{2, \dots, n\}$, construct recursively the game w^k such that $w^k = (w^{k-1})^{\mathbf{N}k}$. By construction, we have the following three properties.

Firstly, for a given $k \in \{2, \dots, n\}$, by definition of a game in which player k is nullified, k is null in game w^k. By Remark 11, we know that player k remains null in game w^q, $q \in \{k+1, \dots, n\}$. Thus, for a given $k \in \{2, \dots, n\}$, player k is null in each game w^q, $q \in \{k, \dots, n\}$. This implies that in w^n, all players $k \in \{2, \dots, n\}$ are null.

Secondly, for all $k \in \{2, \dots, n\}$ and all $S \ni 1$, note that $w^k(S) = (w^{k-1})^{\mathbf{N}k}(S) = w^{k-1}(S \setminus k)$. Since $S \setminus k \ni 1$ and since $w^1(T) = 0$ for all $T \ni 1$, this forces that $w^k(S) = 0$. As a consequence, we get $w^n(S) = 0$ for all $S \ni 1$. Together with the first property, this implies that $w^n = \mathbf{0}$. In a sense, making players $k = \{2, \dots, n\}$ null also eventually makes the nullifying player 1 null. Therefore, $\mathbf{Co}^-$ yields

$$\varphi(w^n) = \varphi(\mathbf{0}) = \mathbf{0}. \tag{8}$$

Thirdly, from the second property, for all $k \in \{2, \dots, n\}$, we have both $w^k(N) = 0$ and $w^k(N \setminus k) = 0$ since these two coalitions contain player 1. Thus, Lemma 14 (a) implies that, for all $k \in \{2, \dots, n\}$,

$$\varphi(w^k) = \varphi((w^{k-1})^{\mathbf{N}k}) = \varphi(w^{k-1}). \tag{9}$$

We are now ready to summarize the findings of all the above steps. For all $j \in N$, combining (5) to (9) yields

$$0 \overset{(8)}{=} \varphi_j(w^n) \overset{(9)}{=} \varphi_j(w^1) \overset{(7)}{=} \varphi_j(v^1) - \frac{v^1(N)}{n} \overset{(6)}{=} \varphi_j(r^1) - \frac{v^1(N)}{n} \overset{(5)}{=} \varphi_j(v) + v(N\setminus 1) - v(N) - \frac{v^1(N)}{n}.$$

Thus, using the fact that $v^1(N) = r^1(N)$ and the definition of r^1, we obtain, for all $j \in N$, that $\varphi_j(v) = v(N)/n = \mathrm{ED}_j(v)$. The proof is complete since v was arbitrarily chosen. □

5. A comparison with van den Brink (2007)

This section deals with another advantage of some of our results over the characterization of the ED-value proposed in Proposition 1 by van den Brink (2007). The result below states that dropping the equal treatment axiom from Proposition 1 does not ensure that the resulting values are weighted division values, even if linearity is imposed instead of additivity.

Proposition 15. *The set of values satisfying efficiency (**E**), additivity (**A**) or linearity (**L**), and the nullifying player axiom (**Nd**) is not contained in $\mathcal{W}$.*

Proof. It suffices to exhibit a value not in $\mathcal{W}$ satisfying **L**, **E**, and **Nf**. For all non-empty $S \subsetneq N$, choose a player denoted by $i(S) \in S$. Define the value φ as

$$\varphi_i(v) = \mathrm{ED}_i(v) + \sum_{S: i(S) = i} v(S) - \sum_{S: S \setminus i(S) \ni i} \frac{v(S)}{s-1} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{10}$$

Since the family $(i(S))_{\emptyset \subsetneq S \subseteq N}$ does not depend on $v \in \mathbb{V}$, φ satisfies **L**. Next, for all non-empty $S \neq N$, $\sum_{i \in N} \varphi_i(e_S) = 0$ and $\sum_{i \in N} \varphi_i(e_N) = 1$. Using (1), conclude that φ satisfies

E. For all v and all $i \in N$, $\varphi_i(v)$ depends only on the worth of coalitions containing player i, so that φ obviously satisfies the nullifying player axiom. The proof is complete since φ cannot belong to $\mathcal{W}$. □

In a sense, the role of the equal treatment axiom in Proposition 1 is not only to assign an identical share of the worth of grand coalition but also to neutralize the influence of all smaller coalitions on the distribution of payoffs. Indeed, we can use two of our results to provide characterizations of the ED-value in which dropping the equal treatment axiom yields the set of positively weighted division values. To understand this aspect, consider the following proposition.

Proposition 16. *The ED-value is the unique value satisfying equal treatment axiom (**ET**) and either*

*(a) efficiency (**E**), additivity (**A**), and the positive player axiom (**PP**).*

*(b) efficiency (**E**), additivity (**A**), and the nullified player axiom (**Nd**).*

Replacing additivity in Proposition 16 by linearity still generates two sets of logically independent axioms (see the appendix for more details). Proceeding in this fashion and dropping the equal treatment axiom as we did in Proposition 1 to obtain Proposition 15, we recover the characterizations of the positively weighted division values provided by Propositions 10 and 12, respectively. Another view on the results in this section is to remark that Propositions 1 and 16 (a) and (b) only differ with respect to one axiom, the nullifying player axiom, the positive player axiom, and the nullified player axiom, respectively. In a sense, the nullifying player axiom is not strong enough to generate only positively weighted division values without the help of the equal treatment axiom as it is the case with the positive player axiom and the nullified player axiom.

Proof. (Proposition 16) We shall only prove the uniqueness parts. For part (a), if **A** replaces **L** and $c \cdot e_S$, $c \in \mathbb{R}$, replaces e_S in the proof of Proposition 10, we still can conclude that $\varphi_i(c \cdot e_S) = 0$ for all $S \neq N$, $i \in N$ and $c \in \mathbb{R}$ since φ remains and odd function. As a consequence, the following representation of any additive value

$$\varphi_i(v) = \sum_{S \subseteq N} \varphi_i(v(S) \cdot e_S) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N$$

can be rewritten as

$$\varphi_i(v) = \varphi_i(v(N) \cdot e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Finally, **E** and **ET** imply $\varphi_i(v(N) \cdot e_N) = v(N)/n$, as desired.

Refering to Proposition 12, the proof part (b) is very much the same as for part (a). □

6. Concluding remarks

We conclude this article with one remark and a recap chart. The reader might wonder whether linearity can be weakened by using additivity in Propositions 4, 9, 10, 12 and Corollary 5, especially because this is exactly what is done in Proposition 16 in a different context. This is not possible. The reason is that there exist additive functions which are not linear, and that linearity cannot be derived from the combination of additivity and the other axioms. As an illustration, let us focus on Proposition 4 in order to show that there are values, outside the set of weighted division values that satisfy additivity, efficiency, addition invariance on bi-partitions, and the nullifying player axiom. As suggested in the proof of Proposition 16, replacing linearity by additivity yields that the value under consideration can be written as

$$\varphi_i(v) = \varphi_i(v(N) \cdot e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Now, fix a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is additive but not linear (Macho-Stadler et al. (2007, p. 352) also consider such a function). Using f, define the non-linear value φ by

$$\varphi_i(v) = \begin{cases} \mathrm{ED}_i(v) + (-1)^i \cdot f(v(N)), & i \in \{1, 2\}, \\ \mathrm{ED}_i(v), & i \in N \setminus \{1, 2\} \end{cases} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Note that f cannot be null everywhere on its domain since otherwise it would be linear. As a consequence, the value φ does not belong to the set of weighted division values even though it satisfies additivity, efficiency, addition invariance on bi-partitions, and the nullifying player axiom.

The characterizations contained in this article are summarized in the following table, in which a "+" means that a value satisfies the axiom, in which "−" has the converse meaning, and in which the "⊕" symbols indicate the axioms used in the corresponding characterization. Also, Propositions and Corollaries are abbreviated by letters P and C respectively, followed by their identifying number.

	$\mathcal{W}$		$\mathcal{W}^+$			ED-value					
	P4	C5	P9	P12	P10	P1	P7	C8	P13	P16a	P16b
Efficiency, **E**	⊕	⊕	⊕	⊕	⊕	⊕	+	+	⊕	⊕	⊕
Equal treatment, **ET**	−	−	−	−	−	⊕	+	+	+	⊕	⊕
Nullifying player, **Nf**	⊕	⊕	⊕	+	+	⊕	⊕	⊕	+	+	+
Additivity, **A**	+	+	+	+	+	⊕	+	⊕	+	⊕	⊕
Linearity, **L**	⊕	⊕	⊕	⊕	⊕	+	+	+	+	+	+
Self-duality, **SD**	+	⊕	+	+	+	+	+	⊕	+	+	+
Addition invariance on bi-partitions, **AIB**	⊕	+	+	+	+	+	⊕	+	+	+	+
Weak covariance, **Co**$^-$	−	−	−	−	−	+	⊕	⊕	⊕	+	+
Null player in a productive environment, **NPE**	−	−	⊕	+	+	+	+	+	+	+	+
Positive player, **PP**	−	−	+	+	⊕	+	+	+	+	⊕	+
Nullified player, **Nd**	−	−	+	⊕	+	+	+	+	⊕	+	⊕

AppendixA. Non-redundancy of characterizations

In this appendix, we show that our characterizations are non-redundant in non-trival cases, *i.e.*, if $|N| > 1$ or $|N| > 2$.

Proposition 4 and Corollary 5

The Shapley value satisfies linearity, efficiency, self-duality, and addition invariance on bi-partitions but not the nullifying player axiom. The null value satisfies linearity, the nullifying player axiom, self-duality, and addition invariance on bi-partitions but not efficiency. The value φ defined by

$$\varphi_i(v) = (v(i) - v(N \setminus i)) \cdot v(N) \quad \text{and} \quad \varphi_1(v) = \left(1 - \sum_{i \in N \setminus 1} [v(i) - v(N \setminus i)]\right) \cdot v(N)$$

for all $v \in \mathbb{V}$ and $i \in N \setminus \{1\}$ satisfies efficiency, self-duality, addition invariance on bi-partitions, and the nullifying player axiom but not linearity. The value φ defined by (10) in the proof of Proposition 15 satisfies linearity, efficiency, the nullifying player axiom but neither self-duality nor addition invariance on bi-partitions.

Proposition 7

The Shapley value satisfies addition invariance on bi-partitions, and weak covariance, but not the nullifying player axiom. The value φ defined by $\varphi_i(v) = v(i)$ for all $v \in \mathbb{V}$ and all $i \in N$ satisfies weak covariance, and the nullifying player axiom, but not addition invariance on bi-partitions. Any value $\varphi \in \mathcal{W}^+ \setminus \{\text{ED}\}$ satisfies addition invariance on bi-partitions, and the nullifying player axiom, but not weak covariance.

Corollary 8

The three values called up just above for Proposition 7 can be used to show that the nullifying player axiom, self-duality, and weak covariance are independent from the other axioms. It remains to show that additivity is independent of the three other axioms. To do so, note that a game v is additive but not symmetric if there exists a weight vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$ with not all identical coordinates and such that $v = \sum_{i \in N} c_i u_i$. Denote by A the class of all games on N that are additive but not symmetric. Define the value φ by

$$\varphi_i(v) = \begin{cases} v(i), & v \in A, \\ \text{ED}_i(v), & v \in \mathbb{V} \setminus A \end{cases} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Note that for all $c, c' \in \mathbb{R}$, $v \in A$ if and only if $(c' \cdot v + \mathbf{c}) \in A$, $c' \neq 0$, *i.e.*, the class of all additive but not symmetric games on N is closed under the "$(c' \cdot v + \mathbf{c})$-operation", provided that $c' \neq 0$. If $c' = 0$ then $(c' \cdot v + \mathbf{c}) = \mathbf{c}$ but in this case, for all $i \in N$, $\text{ED}_i(\mathbf{c}) = c = \mathbf{c}(i)$. As a consequence, φ satisfies weak covariance. For any additive game, observe that $v^D = v$, so that $v \in A$ if and only if $v^D \in A$. In particular, we have $v(i) = v(N) - v(N \setminus i) = v^D(i)$. This implies that φ satisfies self-duality. It is also easy to check that φ satisfies the nullifying player axiom. Finally, consider any game $v \in A$, *i.e.*, $v = \sum_{j \in N} c_j \cdot u_j$ with $c_i \neq c_j$ for some $i, j \in N$. For any given $c \in \mathbb{R} \setminus \{0\}$, both games $c \cdot e_N$ and $v - c \cdot e_N$ are not additive, and thus not in A. It follows that, for all $i \in N$, $\varphi_i(v - c \cdot e_N) = (v(N) - c)/n$ and $\varphi_i(c \cdot e_N) = c/n$. Therefore, $\varphi_i(v - c \cdot e_N) + \varphi_i(c \cdot e_N) = v(N)/n$ for all $i \in N$, *i.e.*, all players get the same payoff in the sum of the two games. But $\varphi_i(v - c \cdot e_N + c \cdot e_N) = \varphi_i(v) = v(i) = c_i$ for all $i \in N$ which implies that not all players get the same payoff in game $v - c \cdot e_N + c \cdot e_N$, proving that φ does not satisfy additivity.

Proposition 9

The Shapley value satisfies linearity, efficiency, and null player in a productive environment axiom but violates the nullifying player axiom. Any value $\varphi \in \mathcal{W} \setminus \mathcal{W}^+$ satisfies linearity, efficiency, the nullifying player axiom but not the null player in a productive environment axiom. The value φ defined by $\varphi_i(v) = v(i)$ for all $v \in \mathbb{V}$ and all $i \in N$ satisfies linearity, the null player in a productive environment axiom, and the nullifying player axiom but not efficiency. The value φ defined by

$$\varphi_i(v) = \begin{cases} \dfrac{v(i)^2}{\sum_{j \in N} v(j)^2} \cdot v(N), & \sum_{j \in N} v(j)^2 \neq 0, \\ \mathrm{ED}_i(v), & \sum_{j \in N} v(j)^2 = 0 \end{cases} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{A.1}$$

satisfies efficiency, the null player in a productive environment axiom, and the nullifying player axiom but not linearity.

Proposition 10

The Shapley value satisfies linearity, efficiency but not the positive player axiom. The null value satisfies linearity, and the positive player axiom but not efficiency. The value given by (A.1) satisfies efficiency and the positive player axiom but not linearity.

Proposition 12

The Shapley value satisfies efficiency and linearity but not the nullified player axiom. The null value satisfies linearity and the nullified player axiom but not efficiency. Consider a vector $\omega \in \mathbb{R}^N$ such that $\sum_{i \in N} \omega_i = 0$ and $\omega_i \neq 0$ for some $i \in N$. The value φ defined by $\varphi_i(v) = \mathrm{ED}_i(v) + \omega_i$ for all $v \in \mathbb{V}$ and $i \in N$ satisfies efficiency and the nullified player axiom but not linearity.

Proposition 13

The Shapley value satisfies efficiency, and weak covariance but not the nullified player axiom. Fix a given player $i \in N$. The value $\varphi^{(i)}$ defined by $\varphi_j^{(i)}(v) = v(i)$ for all $v_i n \mathbb{V}$ and all $j \in N$ satisfies the nullified player axiom and weak covariance but not efficiency. Any value $\varphi \in \mathcal{W}^+ \setminus \{\mathrm{ED}\}$ satisfies efficiency, and the nullified player axiom but not weak covariance.

*Proposition*16 *(a) and (b)*

The Shapley value satisfies efficiency, additivity, and the equal treatment axiom but neither the positive player axiom nor the nullified player axiom. The null value satisfies additivity, the equal treatment axiom, the nullified player axiom, and the positive player axiom but not efficiency. Any value $\varphi \in \mathcal{W}^+ \setminus \{\text{ED}\}$ satisfies efficiency, the nullified player axiom, the positive player axiom, and additivity but not the equal treatment axiom. The value given by (A.1) satisfies efficiency, the positive player axiom, and the equal treatment axiom but not additivity. Suppose that $n \geq 3$. Consider a game $w \in \mathbb{V}$ such that no two distinct players are substitutes,

$$w(N) > 0, \quad \text{and} \quad w(N \setminus i) = 0 \quad \text{for all } i \in N. \tag{A.2}$$

Fix $\omega \in \mathbb{R}^n_+$ such that $\sum_{i \in N} \omega_i = 1$ and $\omega_i \neq \omega_j$ for some $i, j \in N$. Define the value φ by $\varphi_i(w) = \text{WD}^\omega_i(w)$ and $\varphi_i(v) = \text{ED}_i(v)$ if $v \in \mathbb{V} \setminus \{w\}$. Since w does not contain any pair of equal players, φ satisfies the equal treatment axiom. It is also obvious that φ satisfies efficiency. Regarding the nullified player axiom, consider any game $v \in \mathbb{V} \setminus \{w\}$. Since condition (A.2) implies that w does not contain any null player, we have $v^{\mathbf{N}i} \neq w$ for all $i \in N$, so that the nullified player axiom is satisfied when the considered game is $v \in \mathbb{V} \setminus \{w\}$. Now, let us test the nullified player axiom starting with game w. By (A.2), we have $w^{\mathbf{N}i}(N) = w(N \setminus i) = 0$ for all $i \in N$. Therefore,

$$\varphi_i(w) = \text{WD}^\omega_i(w) \geq 0 = \text{ED}_i(w^{\mathbf{N}i}) = \varphi_i(w^{\mathbf{N}i}),$$

but also

$$\varphi_j(w) = \text{WD}^\omega_j(w) \geq 0 = \text{ED}_j(w^{\mathbf{N}j}) = \varphi_j(w^{\mathbf{N}i}),$$

for all $j \in N \setminus i$, which shows that φ satisfies the nullified player axiom. Finally, it is easy to see that φ does not satisfies additivity by considering two games v^1 and v^2 such that $v^1 \neq \mathbf{0}$, $v^2 \neq \mathbf{0}$ and $v^1 + v^2 = w$.

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