An efficient value for TU games with a cooperation structure

resubmitted to the *International Journal of Game Theory* on October 27, 2007

André Casajus*

(August 2006, this version: October 27, 2007, 17:15)

Abstract

We introduce and characterize an efficient value for TU games with a cooperation structure which generalizes the Owen (1977) value for games with a coalition structure but which does not deviate too much from the Myerson (1977) value.

Journal of Economic Literature Classification Number: C71.

Key Words: Efficiency, graph, Owen value, Myerson value.

*Universität Leipzig, Wirtschaftswissenschaftliche Fakultät, Professur für Mikroökonomik, PF 100920, D-04009 Leipzig, Germany, e-mail: casajus@wifa.uni-leipzig.de

I would like to thank the editor of this journal, two anonymous referees, an associate editor, and, last but not least, Frank Huettner for their comments which led to a considerable improvement of the exposition of the ideas in this paper. Of course, the usual disclaimer applies.

1. Introduction

Consider the TU game with the player set $N = \{P1, P2, P3, A\}$ and the coalition function given by

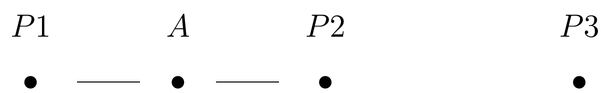
$$v(K) = \begin{cases} 48, & |K \cap \{P1, P2, P3\}| = 3, \\ 36, & |K \cap \{P1, P2, P3\}| = 2, \\ 0, & |K \cap \{P1, P2, P3\}| \leq 1, \end{cases} \qquad , K \subseteq N.$$

There is a Null player, A; any two of the productive players $P1, P2$, and $P3$ can create a worth of 36, while all productive players together produce a worth of 48. Suppose all these players cooperate in order to create the grand coalition's worth of 48. If the players do not form any coalitions when bargaining on the distribution of $v(N)$, then, for symmetry reasons, one would expect an equal split between the three productive players. Would/should this split change if $P1$ and $P2$ formed a bargaining bloc? What if these players could form this bloc only via the Null player A?

As an answer to questions like the first one, Owen (1977) introduces and characterizes an efficient value for games with a coalition structure (partition of the player set); Hart and Kurz (1983; 1984) provide alternative axiomatizations and explore stability issues with respect to the Owen value. In our leading example, the Owen value assigns the payoff 21 to $P1$ and to $P2$ while $P3$ obtains 6 and A gets nothing. Since the players $P1$ and $P2$ together already produce a worth of 36 and since they bargain as one person as well as for symmetry reasons, this fits nicely with our intuitions.

However, the Owen value may not give an adequate answer to the second question. If $P1$ and $P2$ need A in order to form a bargaining bloc then one could argue that—despite being a Null player—A should obtain a positive payoff. But adding A to the bloc formed by $P1$ and $P2$ does not affect the Owen payoffs. The reason for this is that coalition structures are too coarse structures. From the coalition $\{P1, P2, A\}$ alone, one cannot infer whether A is necessary to connect the productive players $P1$ and $P2$ or not. The necessity of A can be modelled by the undirected graph

(1.1)



where $P1$ and $P2$ are connected, but only via a chain of links involving A. Of course, this transcends the world of coalition structures and leads into the realm of cooperation structures (undirected graphs).

There is a number of values for TU games with a cooperation structure (henceforth CO-games and CO-values). The most eminent of these concepts certainly is the value introduced by Myerson (1977). Alternatively, Meessen (1988) suggests the position value which has been popularized by Borm, Owen and Tijs (1992); Bilbao, Jiménez and López (2006) discuss the Hamiache (1999) CO-value. All these CO-values, however, have in common that they are component efficient, which—in contrast to efficiency—corresponds to the interpretation that connected components are the productive units. Vázquez-Brage, García-Jurado and Carreras (1996) suggest a generalization of both the Owen value and the Myerson value, but which refers to games that come with both a coalition structure and a cooperation structure. As the values above, this value is *component efficient* with respect to the cooperation structure (see Section 5).

In our leading example, the Myerson payoffs for the graph in (1.1) are as follows: Since $P3$ is isolated and since he is not productive on his own, $P3$ obtains a zero payoff; and since $P1$ and $P2$ are productive only if they are connected via A, they equally split the worth of their component, $v(\{P1, P2, A\}) = 36$. Hence, the Myerson value recognizes the coordinating role of player A in the situation above. Yet, this value does not account for the productive role of player $P3$ within the grand coalition. Of course, this is a consequence of the Myerson value being component efficient. Hence, one would like to have an *efficient* CO-value which combines properties of the Owen value and of the Myerson value.

Therefore, we introduce and characterize a CO-value that generalizes the Owen value to the class of CO-games and which, in a sense, does not deviate too much from the Myerson value. More specifically, our CO-value coincides with the Owen value for internal completely connected components and coincides with the Myerson value for connected graphs. For the graph (1.1) in our leading example, that CO-value assigns the payoffs $\varphi_{P1} = \varphi_{P2} = 20$, $\varphi_A = 2$, and $\varphi_{P3} = 6$ which meet our intuitions concerning the coordinating role of player A as well as concerning the productive role of player $P3$.

Our characterization involves four axioms. Besides efficiency, we require our CO-value to assign the same payoffs for the complete graph and for the empty graph. Further, merging connected components into single players should not affect the components' payoffs. Finally, we modify the Myerson fairness axiom such that the number of components involved is not affected by removing a link, while the player set involved may shrink.

The plan of this paper is as follows: Basic definitions and notation are given in second section. In the third section, we discuss some axioms related to CO-values. Our CO-value is introduced and axiomatized in the fourth section. The fifth section explores the relation to other CO-values. The payoffs of Null players are considered in the six' section. A few remarks conclude the paper.

2. Basic definitions and notation

Let $\mathcal{U}$ be an infinite set. A **(TU) game** is a pair (N, v) consisting of a non-empty and finite set $N \subseteq \mathcal{U}$, the **player set**, and a function $v \in V(N) := \left\{f : 2^N \to \mathbb{R} | v(\emptyset) = 0\right\}$, the **coalition function**. In general, we consider the set of all TU games, possibly equipped with some additional structure. Subsets of N are called coalitions; $v(K)$ is called the worth of $K \subseteq N$. For $v, v' \in V(N)$, $v + v' \in V(N)$ is given by $(v + v')(K) := v(K) + v'(K)$ for all $K \subseteq N$; $v|_{N'}$ denotes the restriction of v to $N' \subseteq N$. A game is called **strictly convex** iff $v(K \cup K') + v(K \cap K') > v(K) + v(K')$ for all $K, K' \subseteq N$ such that $K \nsubseteq K'$ and $K' \nsubseteq K$. A player i is called a **Null player** iff $v(K) = v(K \cup \{i\})$ for all $K \subseteq N \backslash \{i\}$.

A **value** φ assigns payoff vectors $\varphi(N, v) \in \mathbb{R}^N$ to all games. Set $\Sigma(N) := \{\sigma : N \to \{1, \ldots, |N|\} | \sigma \text{ is bijective}\}$. For $i \in N$ and $\sigma \in \Sigma(N)$, let $K_i(\sigma) := \{j \in N | \sigma(j) \leq \sigma(i)\}$ and $MC_i^v(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley (1953) value Sh is defined by

$$\text{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma), \quad i \in N. \tag{2.1}$$

Let $\mathbb{P}(N)$ denote the set of all partitions of N. A **coalition structure** for (N, v) is some $\mathcal{P} \in \mathbb{P}(N)$; $\mathcal{P}(i)$ denotes the **component** containing player i; $\langle N \rangle := \{\{i\} | i \in N\}$. A **CS-game** is a triple $(N, v, \mathcal{P})$, where (N, v) is a game and $\mathcal{P} \in \mathbb{P}(N)$; a **CS-value** φ assigns payoff vectors $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$ to all CS-games. For $\mathcal{P} \in \mathbb{P}(N)$, we set

$$\Sigma(N, \mathcal{P}) := \{\sigma \in \Sigma(N) | \forall P \in \mathcal{P} \wedge \forall i, j \in P : |\sigma(i) - \sigma(j)| < |P|\}. \tag{2.2}$$

The **Owen (1977) value** is given by

$$\text{Ow}_i(N, v, \mathcal{P}) := |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_i^v(\sigma), \quad i \in N. \tag{2.3}$$

Any $\sigma \in \Sigma(N,\mathcal{P})$ uniquely determines some $\sigma|_{\mathcal{P}} \in \Sigma(\mathcal{P})$ and $\sigma|_P \in \Sigma(P)$, $P \in \mathcal{P}$ as follows: If $P, Q \in \mathcal{P}$ such that $\sigma(i) < \sigma(j)$ for all $i \in P$ and $j \in Q$, then $\sigma|_{\mathcal{P}}(\mathcal{P}(i)) < \sigma|_{\mathcal{P}}(\mathcal{P}(j))$; $\sigma|_P(i) < \sigma|_P(j)$ iff $\sigma(i) < \sigma(j)$ for all $i, j \in P$. For $\tau \in \Sigma(\mathcal{P}(i))$ and $\rho \in \Sigma(\mathcal{P})$, we set

$$\Sigma_i(N,\mathcal{P},\tau,\rho) := \left\{\sigma \in \Sigma(N,\mathcal{P}) \middle| \sigma|_{\mathcal{P}} = \rho \wedge \sigma|_{\mathcal{P}(i)} = \tau\right\}. \tag{2.4}$$

A **cooperation structure** for (N,v) is an undirected graph (N,L), $L \subseteq L^N := \{\{i,j\} \,|\, i,j \in N, i \neq j\}$; a typical element of L is written as ij. Given any graph (N,L), N splits into (maximal connected) **components** the set of which is denoted by $\mathcal{C}(N,L) \in \mathbb{P}(N)$; $C_i(N,L) \in \mathcal{C}(N,L)$ denotes the component containing $i \in N$. The graph (N,L) is called **connected** if $|\mathcal{C}(N,L)| = 1$. Set $L|_S = \{ij \in L | i,j \in S\}$, $S \subseteq N$. For $\mathcal{P} \in \mathbb{P}(N)$, set $L^{\mathcal{P}} := \bigcup_{P \in \mathcal{P}} L^P$, which implies $\mathcal{C}\left(N, L^{\mathcal{P}}\right) = \mathcal{P}$. A **CO-game** is a triple (N,v,L), where (N,v) is a game and $L \subseteq L^N$; a **CO-value** assigns payoff vectors $\varphi(N,v,L) \in \mathbb{R}^N$ to all CO-games. The **Myerson (1977) value** μ is defined by

$$\mu(N,v,L) := \mathrm{Sh}\left(N, v^L\right), \qquad v^L(K) := \sum_{S \in \mathcal{C}(K, L|_K)} v(S), \quad K \subseteq N. \tag{2.5}$$

3. Axioms for CO-values

In this section, we consider several axioms for CO-values with respect to bargaining within the grand coalition.

Efficiency, E. $\sum_{i \in N} \varphi_i(N,v,\cdot) = v(N)$.

The efficiency axiom presupposes the grand coalition to be the productive unit which creates its worth $v(N)$. This corresponds to the interpretation of the connected components of L as bargaining blocs which are formed via bilateral agreements or communication channels.

Component efficiency, CE. For all $C \in \mathcal{C}(N,L)$, we have

$$\textstyle\sum_{i \in C} \varphi_i(N,v,\cdot) = v(C).$$

Component efficiency evokes another interpretation of the graph L. In order to cooperate in the production of worth, players have to be connected via a chain of links. Hence, the connected components C of L are the productive units which produce their worth $v(C)$, respectively.

Fairness, F. For all $ij \in L$, we have

$$\varphi_i(N, v, L) - \varphi_i(N, v, L \backslash \{ij\}) = \varphi_j(N, v, L) - \varphi_j(N, v, L \backslash \{ij\}).$$

CE and **F** are the original axioms that characterize the Myerson value. The very nice fairness axiom **F** has strong consequences far beyond pure fairness considerations. In particular, van den Nouweland (1993, pp. 28) shows that μ satisfies the following axiom which says that (the distribution of) payoffs within a component is not affected by the players outside. In general, of course, **CD** and **E** are incompatible.

Component decomposability, CD. For all $i \in N$,

$$\varphi_i(N, v, L) = \varphi_i\left(C_i(N, L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right).$$

Equivalence, Q. $\varphi\left(N, v, L^N\right) = \varphi(N, v, \emptyset)$.

Axiom **Q** says that—from a bargaining viewpoint—it does not make a difference whether the players do not form any bargaining blocs ($L = \emptyset$) or whether they form just one such bloc where all players are completely connected ($L = L^N$). This seems to be justified because in $\left(N, v, L^N\right)$ as well as in $(N, v, \emptyset)$ all players are symmetric with respect to the graph. Hence in both cases, the payoffs should be determined by the underlying TU game only. Similarly, the Owen payoffs for $\mathcal{P} = \{N\}$ and $\mathcal{P} = \langle N \rangle$ coincide. We feel that **Q** as a natural generalization of this property should be satisfied by an efficient CO-value.

Within the next two axioms, we employ the following definitions: Let (N, v) be a game, $\mathcal{P} \in \mathbb{P}(N)$, and $\pi : \mathcal{P} \to \mathcal{U}$ be an injection. Then, the game $(\pi(\mathcal{P}), \pi v)$ is defined by

$$\pi v(S) := v\left(\bigcup_{k \in S} \pi^{-1}(k)\right), \qquad S \subseteq \pi(\mathcal{P}). \tag{3.1}$$

Component merging, CM. Let $\pi : \mathcal{C}(N, L) \to \mathcal{U}$ be injective. For all $C \in \mathcal{C}(N, L)$, we have

$$\sum_{i \in C} \varphi_i(N, v, L) = \varphi_{\pi(C)}(\pi(\mathcal{C}(N, L)), \pi v, \emptyset).$$

Technically, this axiom says that the components' payoffs can be determined within the game between separated representatives of the components, where the representatives are players from $\mathcal{U}$ who are selected by some mapping π. Axiom **CM** thus is tantamount to that merging components into single players does not affect the components' payoffs, i.e., the inner structure of the components does not matter in this respect. This seems to be plausible as long as the players do not know about the structure of the other components. Otherwise, it could matter how strongly a component is connected. For example, a completely connected component is less likely to split up than a component where there is a single player who keeps the component together. Since the empty graph corresponds to the atomistic partition, **CM** can be viewed as a generalization of Owen's (1977) axiom **A3** below.

Owen's A3. Let $\pi : \mathcal{P} \to \mathcal{U}$ be injective. For all $P \in \mathcal{P}$, we have

$$\sum_{i \in P} \varphi_i \left(N, v, \mathcal{P}\right) = \varphi_{\pi(P)} \left(\pi\left(\mathcal{P}\right), \pi v, \langle \pi\left(\mathcal{P}\right)\rangle\right).$$

Modified fairness, MF. For all $ij \in L$,

$$\varphi_i \left(N, v, L\right) - \varphi_i \left(N_i \left(L, ij\right), v|_{N_i(L,ij)}, L \backslash \{ij\} |_{N_i(L,ij)}\right) = \varphi_j \left(N, v, L\right) - \varphi_j \left(N_j \left(L, ij\right), v|_{N_j(L,ij)}, L \backslash \{ij\} |_{N_j(L,ij)}\right)$$

where

$$N_i \left(L, ij\right) := N \backslash \left(C_i \left(N, L\right) \backslash C_i \left(N, L \backslash \{ij\}\right)\right). \tag{3.2}$$

As **F**, this axiom is concerned with the effects of removing the link ij on the payoffs of i and j. **MF** requires that the gains/losses of the players involved with respect to some reference game should be equal. Within **F**, the reference game is just the original game without the removed link, $(N, v, L \backslash \{ij\})$. This way, the player set remains unchanged, but the number of components may increase, i.e., **F** compares the payoffs of games with possibly different numbers of components. Since **F** is employed in conjunction with **CE**, the components can be viewed as productive units which may or may not interact. However, the Myerson value, which is characterized by **F** and **CE**, also satisfies **CD**, i.e., the productive units do not interact. Therefore, the number of components is irrelevant.

Since we are concerned with an efficient value where the components are viewed as bargaining blocs, the number of components may be crucial. Therefore, we suggest **MF** as a modification of **F** which involves reference games with the same number of components as the original one. If the removal of ij does not split $C_i(N, L) = C_j(N, L)$ then (3.2) implies $N_i(L, ij) = N_j(L, ij) = N$. In this case, **MF** coincides with **F**. Otherwise, $C_i(N, L)$ splits into two disjoint components. In order to keep the number of components in the reference game, it seems to be the obvious choice to remove those players from consideration who were connected (in the same component) with, say, i, but are no longer connected with i after cutting the link ij. This is exactly what (3.2) does. We then have $N_i(L, ij) = N \backslash C_j(N, L \backslash \{ij\})$ and $N_j(L, ij) = N \backslash C_i(N, L \backslash \{ij\})$. In a sense, **MF** is the natural modification of **F** if one wishes to compare games with the same number of components.

4. A generalization of the Owen value

In this section, we show that **E**, **Q**, **MF**, and **CM** already characterize a CO-value. We first deal with uniqueness and then with existence.

If the graph of a CO-game is connected, then all players are contained in one bargaining bloc. In this case, one could argue that the distribution of the grand coalition's worth should be governed by the inner structure of that single bloc and by the fairness considerations embodied in the Myerson value. Yet, this already is implied by **E** and **MF**.

Lemma 4.1. *If a CO-value φ satisfies **E** and **MF** then it coincides with μ on all connected graphs.*

Proof. We first note that for connected graphs **MF** involves connected graphs only. μ satisfies **CE** which for connected graphs becomes **E**. We also have

$$\begin{aligned}
&\mu_i(N, v, L) - \mu_j(N, v, L) \\
&\qquad \overset{\mathbf{F}}{=} \mu_i(N, v, L \backslash \{ij\}) - \mu_j(N, v, L \backslash \{ij\}) \\
&\qquad \overset{\mathbf{CD}}{=} \mu_i\left(C_i(N, L \backslash \{ij\}), v|_{C_i(N,L\backslash\{ij\})}, L \backslash \{ij\}|_{C_i(N,L\backslash\{ij\})}\right) \\
&\qquad\qquad - \mu_j\left(C_j(N, L \backslash \{ij\}), v|_{C_j(N,L\backslash\{ij\})}, L \backslash \{ij\}|_{C_j(N,L\backslash\{ij\})}\right) \\
&\qquad \overset{\mathbf{CD}}{=} \mu_i\left(N_i(L, ij), v|_{N_i(L,ij)}, L \backslash \{ij\}|_{N_i(L,ij)}\right) \\
&\qquad\qquad - \mu_j\left(N_j(L, ij), v|_{N_j(L,ij)}, L \backslash \{ij\}|_{N_j(L,ij)}\right)
\end{aligned}$$

for $ij \in L$ where the third equation follows from $C_i(N, L \backslash \{ij\}) = C_i(N_i(L, ij), L \backslash \{ij\}|_{N_i(L,ij)})$ and the analogon for j. Hence, μ satisfies **MF**.

We mimic the Myerson (1977) proof of uniqueness. Suppose both φ and $\bar{\varphi}$ satisfy **E** and **MF**. Suppose N is a minimal player set such that φ and $\bar{\varphi}$ differ on a connected graph. Further, suppose L is a minimal connected graph on N such that they do so. By **E** and since L is connected, L contains at least one link. If $j \in C_i(N, L \backslash \{ij\})$ then **MF** and the minimality of L imply $\varphi_i(N, v, L) - \varphi_j(N, v, L)$

$= \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. And if $j \notin C_i(N, L\backslash\{ij\})$ then again **MF** and the minimality of N imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. Since L is connected, we have $\varphi_i(N,v,L) - \bar{\varphi}_i(N,v,L) = \Delta$ for some $\Delta \in \mathbb{R}$ and all $i \in N$. **E** then implies $\Delta = 0$. Contradiction. ∎

Applying this Lemma and again the Myerson technique, we are now able to approach the general case.

Theorem 4.1. *There is at most one CO-value that satisfies **E**, **Q**, **MF**, and **CM**.*

In view of their role in the proof below, one could of course merge **Q** and **CM** into a single axiom. However, we feel that the two axioms refer to essentially different considerations. While **Q** basically is a very weak expression of invariance with respect to the renaming of players, **CM** requires the payoff of the components to be independent of their inner structure.

Proof. Let φ be a CO-value that satisfies **E**, **Q**, **MF**, and **CM**. Let $\pi : \mathcal{C}(N,L) \to \mathcal{U}$ be injective. By **CM** and **Q**, we have

$$\sum_{i \in C} \varphi_i(N,v,L) = \varphi_{\pi(C)}\left(\pi(\mathcal{C}(N,L)), \pi v, L^{\pi(\mathcal{C}(N,L))}\right)$$

for all $C \in \mathcal{C}(N,L)$. Since $\left(\pi(\mathcal{C}(N,L)), L^{\pi(\mathcal{C}(N,L))}\right)$ is connected, Lemma 4.1 implies

$$\sum_{i \in C} \varphi_i(N,v,L) = \mu_{\pi(C)}\left(\pi(\mathcal{C}(N,L)), \pi v, L^{\pi(\mathcal{C}(N,L))}\right). \tag{4.1}$$

Again, we mimic the Myerson (1977) proof of uniqueness. Suppose there were two CO-values, φ and $\bar{\varphi}$, that satisfy **E**, **Q**, **MF**, and **CM**. Let L be a minimal player set such that $\varphi \neq \bar{\varphi}$ and let L be a minimal graph on N such that they do so. By **E**, N then contains more than one player, and by **Q** and Lemma 4.1, L contains at least one link. If $C_i(N,L) = \{i\}$, then $\varphi_i(N,v,L) = \bar{\varphi}_i(N,v,L)$ by (4.1). For $ij \in L$, we have

$$\begin{aligned}
\varphi_i(N,v,L) - \varphi_j(N,v,L) &= \varphi_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L\backslash\{ij\}|_{N_i(L,ij)}\right) \\
&\quad - \varphi_j\left(N_j(L,ij), v|_{N_j(L,ij)} L\backslash\{ij\}|_{N_j(L,ij)}\right) \\
&= \bar{\varphi}_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L\backslash\{ij\}|_{N_i(L,ij)}\right) \\
&\quad - \bar{\varphi}_j\left(N_j(L,ij), v|_{N_j(L,ij)} L\backslash\{ij\}|_{N_j(L,ij)}\right) \\
&= \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)
\end{aligned}$$

by **MF**, the minimality of N and L, and again **MF**. Thus, we have $\varphi_j(N,v,L) - \bar{\varphi}_j(N,v,L) = \varphi_k(N,v,L) - \bar{\varphi}_k(N,v,L)$ for all $j,k \in C_i(N,L)$. In view of (4.1), this implies $\varphi_j(N,v,L) = \bar{\varphi}_j(N,v,L)$ for all $j \in C_i(N,L)$. Contradiction. ∎

Theorem 4.2. *The CO-value φ given by*

$$\varphi_i(N,v,L) = |\Sigma(\mathcal{C}(N,L))|^{-1} \sum_{\rho \in \Sigma(\mathcal{C}(N,L))} \mu_i\left(C_i(N,L), v^{\rho}_{C_i(N,L)}, L|_{C_i(N,L)}\right) \tag{4.2}$$

where v^{σ}_C is given by

$$v^{\rho}_C(S) = v\left(K^-_C(\rho) \cup S\right) - v\left(K^-_C(\rho)\right), \qquad S \subseteq C, \tag{4.3}$$

where $K^-_C(\rho) := \bigcup_{C' \in \mathcal{P}:\rho(C')<\rho(C)} C'$ for all $C \in \mathcal{C}(N,L)$ and $\rho \in \Sigma(\mathcal{C}(N,L))$ satisfies ***E****,* ***Q****,* ***MF****, and* ***CM****.*

The intuition behind (4.2) and (4.3) is the following: We think of the players entering a room in some order compatible with the partition $\mathcal{C}(N,L)$. On the one hand, this is reflected by considering all orders ρ of $\mathcal{C}(N,L)$. On the other hand, the Myerson value is applied to the components. Yet, by (2.5) and (2.1), this value involves all orders of the players in the component under consideration. By (2.5), the application of the Myerson value implies that for any such order the players are rewarded with their marginal contributions in the graph restricted game. If the coalition function were not modified by (4.3) then the resulting value would be component efficient, but not necessarily efficient. The modifications of the coalition function in (4.3) model the fact that *all* players cooperate in order to produce the grand coalition's worth. More specifically, upon entrance, the players of some component $C \in \mathcal{C}(N,L)$ consider the players of those components who entered the room before them according to ρ, i.e., the players in $K^-_C(\rho)$, to be present with respect to the production of worth. Note that by (2.5), this would not be the case without the modification. Hence, our CO-value combines the Owen value, which controls the distribution of $v(N)$ between the components, and the Myerson value, which controls the distribution of $\sum_{i\in C}\varphi_i(N,v,L)$ within the components C taking into account the cooperation structure L.

Proof. Consider the CO-value φ as in (4.2). We then have

$$\begin{aligned}\sum_{i\in N}\varphi_i(N,v,L) &= |\Sigma(\mathcal{C}(N,L))|^{-1}\sum_{\rho\in\Sigma(\mathcal{C}(N,L))}\sum_{C\in\mathcal{C}(N,L)}\mu_C\left(C,v_C^{\rho},L|_C\right)\\ &= |\Sigma(\mathcal{C}(N,L))|^{-1}\sum_{\rho\in\Sigma(\mathcal{C}(N,L))}\sum_{C\in\mathcal{C}(N,L)}v_C^{\rho}(C)\\ &= |\Sigma(\mathcal{C}(N,L))|^{-1}\sum_{\rho\in\Sigma(\mathcal{C}(N,L))}v(N)=v(N)\end{aligned}$$

by (4.2) and changing the order of summation, by the fact that $(C,L|_C)$ is connected for $C\in\mathcal{C}(L,N)$ and that μ is component efficient, and by (4.3). Hence, φ satisfies **E**.

For $L=\emptyset$, we have $C_i(N,L)=\{i\}$, hence $\mathcal{C}(N,L)\cong N$ and $\Sigma(\mathcal{C}(N,L))\cong\Sigma(N)$, and therefore

$$\begin{aligned}\varphi_i(N,v,\emptyset) &= |\Sigma(\mathcal{C}(N,L))|^{-1}\sum_{\rho\in\Sigma(C(N,L))}\mu_i\left(\{i\},v_i^{\rho},\emptyset\right)\\ &= |\Sigma(N)|^{-1}\sum_{\sigma\in\Sigma(N)}v_i^{\sigma}(i)\\ &= |\Sigma(N)|^{-1}\sum_{\sigma\in\Sigma(N)}MC_i^v(\sigma)=\mathrm{Sh}_i(N,v)\end{aligned}$$

by (4.2), again by the fact that $(\{i\},\emptyset)$ is connected and that μ is component efficient, by (4.3) and the definition of $MC_i^v(\sigma)$, and by (2.1). Further, by (4.2) and (4.3), we have $\varphi\left(N,v,L^N\right)=\mu\left(N,v,L^N\right)$ since $\left(N,L^N\right)$ is connected. Since $\mu\left(N,v,L^N\right)=\mathrm{Sh}(N,v)$ (Myerson, 1977), φ also satisfies **Q**.

Next, we show that φ satisfies **CM**. Let $\pi:\mathcal{C}(N,L)\rightarrow\mathcal{U}$ be injective. In view of (4.2), it suffices to show that

$$\sum_{i\in C}\mu_i\left(C,v_C^{\rho},L|_C\right)=\mu_{\pi(C)}\left(\{\pi(C)\},\pi v_{\{\pi(C)\}}^{\rho\circ\pi},\emptyset\right)$$

for all $C\in\mathcal{C}(N,L)$ and $\rho\in\Sigma(\mathcal{C}(N,L))$. Since μ is component efficient and $(C,L|_C)$ as well as $(\{\pi(C)\},\emptyset)$ are connected for $C\in\mathcal{C}(N,L)$ in general, this is equivalent to $v_C^{\rho}(C)=\pi v_{\{\pi(C)\}}^{\rho\circ\pi}(\{\pi(C)\})$ which holds by (4.3) and (3.1).

Finally, we show that φ satisfies **MF**. In view of (4.2) and (4.3), it suffices to show that

$$\begin{aligned}
&\mu_i\left(C_i(N,L), v^{\rho}_{C_i(N,L)}, L|_{C_i(N,L)}\right) \\
&\quad -\mu_i\left(C_i(N,L\backslash\{ij\}), v^{\rho}_{C_i(N,L)}|_{C_i(N,L\backslash\{ij\})}, L\backslash\{ij\}|_{C_i(N,L\backslash\{ij\})}\right) \\
&\qquad = \mu_j\left(C_j(N,L), v^{\rho}_{C_j(N,L)}, L|_{C_j(N,L)}\right) \\
&\qquad\quad -\mu_j\left(C_j(N,L\backslash\{ij\}), v^{\rho}_{C_j(N,L)}|_{C_j(N,L\backslash\{ij\})}, L\backslash\{ij\}|_{C_j(N,L\backslash\{ij\})}\right)
\end{aligned}$$

holds for all $ij \in L$ and $\rho \in \Sigma(\mathcal{C}(N,L))$, where ρ is interpreted as a member of $\Sigma\left(\mathcal{C}\left(N_k(L,ij), L|_{N_k(L,ij)}\right)\right)$, $k = i,j$ in the obvious way. Yet, this follows from μ satisfying **F** and **CD**. ∎

5. Relation to other values

Below, we show that the value defined by (4.2) and (4.3) is a generalization of the Owen value. This justifies the notation $\mathrm{Ow}^{\#}$ where the symbol # is intended to indicate a graph. From (4.2) and (4.3) it is easy to see that $\mathrm{Ow}^{\#}$ and μ coincide on connected graphs and that $\mathrm{Ow}^{\#}$ inherits additivity—$\varphi(N, v+v', L) = \varphi(N,v,L) + \varphi(N,v',L)$—from the Myerson value in general. The characterizations of the Owen value by Owen (1977) himself as well as those of Hart and Kurz (1983) involve the additivity axiom. Khmelnitskaya and Yanovskaya (2007) characterize the Owen value without additivity. In view of the following theorem, our value indeed extends the Owen value to CO-games and therefore provides another justification of the Owen value without the additivity axiom.

Theorem 5.1. $\mathrm{Ow}^{\#}\left(N,v,L^{\mathcal{P}}\right) = \mathrm{Ow}(N,v,\mathcal{P})$.

Proof. Since $C_i\left(N,L^{\mathcal{P}}\right) = \mathcal{P}(i)$ for all $i \in N$, we have

$\mathrm{Ow}^{\#}_i\left(N,v,L^{\mathcal{P}}\right)$

$$\overset{(4.2)}{=} \quad |\Sigma(\mathcal{P})|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} \mu_i \left(\mathcal{P}(i), v^{\rho}_{\mathcal{P}(i)}, L^{\mathcal{P}(i)} \right)$$

$$\overset{(2.5)}{=} \quad |\Sigma(\mathcal{P})|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} \mathrm{Sh}_i \left(\mathcal{P}(i), v^{\rho}_{\mathcal{P}(i)} \right)$$

$$\overset{(2.1)}{=} \quad |\Sigma(\mathcal{P})|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} |\Sigma(\mathcal{P}(i))|^{-1} \sum_{\tau \in \Sigma(\mathcal{P}(i))} MC_i^{v^{\rho}_{\mathcal{P}(i)}}(\tau)$$

$$\overset{(4.3),(2.4)}{=} \quad |\Sigma(\mathcal{P})|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} |\Sigma(\mathcal{P}(i))|^{-1} \sum_{\tau \in \Sigma(\mathcal{P}(i))} |\Sigma_i(N, \mathcal{P}, \tau, \rho)|^{-1} \sum_{\sigma \in \Sigma_i(N, \mathcal{P}, \tau, \rho)} MC_i^v(\sigma)$$

$$\overset{(2.4)}{=} \quad |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_i^v(\sigma)$$

$$\overset{(2.3)}{=} \quad \mathrm{Ow}_i(N, v, \mathcal{P})$$

for all $i \in N$. ∎

Since the Owen value and the Shapley value coincide for $\mathcal{P} = \{N\}$ and $\mathcal{P} = \langle N \rangle$ the following property is immediate.

Corollary 5.1. $\mathrm{Ow}^{\#}\left(N, v, L^N\right) = \mathrm{Ow}^{\#}(N, v, \emptyset) = \mathrm{Sh}(N, v)$.

Finally, **CM** and **Q** then imply that the distribution of the grand coalition's worth between components is governed by the same principles for $\mathrm{Ow}^{\#}$ and Ow.

Corollary 5.2. *For all* $C \in \mathcal{C}(N, L)$,

$$\sum_{i \in C} \mathrm{Ow}_i^{\#}(N, v, L) = \sum_{i \in C} \mathrm{Ow}_i(N, v, \mathcal{C}(N, L)).$$

Vázquez-Brage et al. (1996) suggest a generalization of both the Owen value and the Myerson value. In particular, they consider TU games that come with both a cooperation structure L and a coalition structure $\mathcal{P}$. Their value ψ is given by

$$\psi(N, v, L, \mathcal{P}) := \mathrm{Ow}\left(N, v^L, \mathcal{P}\right), \tag{5.1}$$

which gives $\psi\left(N, v, L^N, \mathcal{P}\right) = \mathrm{Ow}(N, v, \mathcal{P})$ and $\psi(N, v, L, \{N\}) = \psi(N, v, L, \langle N \rangle)$ $= \mu(N, v, L)$.

Then, one might wonder whether the $\mathrm{Ow}^{\#}$-value also can be expressed as the ψ-value with suitable choices of the graph and of the partition. In view of (2.3), (4.2), and (5.1), the natural choice for $\mathcal{P}$ is $\mathcal{C}(N, L)$. But what about the graph? Since ψ is component efficient on the graph, i.e., $\sum_{i \in C} \psi_i(N, v, L, \mathcal{P}) = v(C)$ for all $C \in \mathcal{C}(N, L)$, while $\mathrm{Ow}^{\#}$ is efficient, one has to construct a connected graph $\Lambda(L)$

from L which preserves the inner structure of the components of L. For symmetry reasons, the graph $\Lambda(L)$ given by

$$\Lambda(L) := L \cup \{ij \in L^N | C_i(N,L) \neq C_j(N,L)\} \tag{5.2}$$

is the natural candidate. Set $\mathrm{Ow}^{\psi}(N,v,L) := \mathrm{Ow}\left(N, v^{\Lambda(L)}, \mathcal{C}(N,L)\right)$. Yet, from (5.2) and (2.5) it is clear that $v^{\Lambda(L)}(K) = v(K)$ whenever $K \nsubseteq P$ for some $P \in \mathcal{C}(N,L)$. Comparing this with (4.2) and (4.3) already indicates that Ow^{ψ}—in contrast to $\mathrm{Ow}^{\#}$—does not exploit the inner structure of the components on the whole domain of v.

Moreover, our leading example reveals that $\mathrm{Ow}^{\#}$ and Ow^{ψ} do not coincide in general. From the graph L in (1.1), one derives $\Lambda(L) = L^N \backslash \{P1P2\}$ and then obtains

$$v^{\Lambda(L)}(K) = \begin{cases} 48, & |K \cap \{P1,P2,P3\}| = 3, \\ 36, & |K \cap \{P1,P2,P3\}| = 2 \wedge K \neq \{P1,P2\}, \\ 0, & |K \cap \{P1,P2,P3\}| \leq 1 \vee K = \{P1,P2\}, \end{cases} \qquad K \subseteq N.$$

By (2.3), this gives the payoffs $\mathrm{Ow}_i^{\psi} := \mathrm{Ow}_i^{\psi}(N,v,L)$, $i \in N$, $\mathrm{Ow}_{P1}^{\psi} = \mathrm{Ow}_{P2}^{\psi} = 18$ and $\mathrm{Ow}_{P3}^{\psi} = \mathrm{Ow}_{A}^{\psi} = 6$. Since these payoffs differ from the $\mathrm{Ow}^{\#}$-payoffs in Section , the ψ-value and the $\mathrm{Ow}^{\#}$-value are essentially different.

6. Null players

Reconsider our leading example with the graph in (1.1). There are two orders, σ and ρ, on $\mathcal{C}(N,L) = \{C, \{P3\}\}$, $C = \{P1, P2, A\}$ where $\sigma(C) = 1$ and $\rho(C) = 2$. By (4.3), this gives the payoff functions $v_C^{\sigma}(S) = v(S)$, $S \subseteq C$ and

$$v_C^{\rho}(S) = \begin{cases} 48, & |S \cap \{P1,P2\}| = 2, \\ 36, & |S \cap \{P1,P2\}| = 1, \\ 0, & S = \emptyset \vee S = \{A\}, \end{cases} \qquad S \subseteq C.$$

Applying (2.5) yields the payoffs $\mu_i^{\sigma} := \mu_i(C, v_C^{\sigma}, L|_C)$ and $\mu_i^{\rho} := \mu(C, v_C^{\rho}, L|_C)$, $i \in C$ as follows: $\mu_{P1}^{\sigma} = \mu_{P2}^{\sigma} = \mu_A^{\sigma} = 12$, $\mu_{P1}^{\rho} = \mu_{P2}^{\rho} = 28$, and $\mu_A^{\rho} = -8$. By (4.2), this gives the payoffs $\mathrm{Ow}_i^{\#} := \mathrm{Ow}_i^{\#}(N,v,L)$, $i \in N$, $\mathrm{Ow}_{P1}^{\#} = \mathrm{Ow}_{P2}^{\#} = 20$, $\mathrm{Ow}_{P2}^{\#} = 6$, and $\mathrm{Ow}_A^{\#} = 2$ as in Section 1, where the payoff for $P3$ is immediate from **E**.

The leading example demonstrates that a Null player's $\mathrm{Ow}^{\#}$-payoff may be positive if he facilitates the formation of an advantageous bargaining bloc. The theorem below generalizes this observation: Under the hypothesis of the theorem, on the one hand, there are strictly positive gains from cooperation among the other players, and on the other hand, the Null player is crucial for forming one of the components.

Theorem 6.1. *If i is a Null player in (N,v), $\left(N\backslash\{i\}, v|_{N\backslash\{i\}}\right)$ is strictly convex, and $|\mathcal{C}(N,L)| < \left|\mathcal{C}\left(N\backslash\{i\}, L|_{N\backslash\{i\}}\right)\right|$ then* $\mathrm{Ow}_i^{\#}(N,v,L) > 0$.

Proof. Let (N,v,L) and $i \in N$ be as in the theorem. Set $C := C_i(N,L)$. In view of (4.2) and (4.3), it suffices to show that $\mu_i(C, v_C^{\rho}, L|_C) > 0$ for all $\rho \in \Sigma(\mathcal{C}(N,L))$. Since

$$\mu_i(C, v_C^{\rho}, L|_C) = \mathrm{Sh}_i\left(C, (v_C^{\rho})^{L|_C}\right) = |\Sigma(N)|^{-1} \sum_{\sigma\in\Sigma(C)} MC_i^{\left(v_C^{\rho}\right)^{L|_C}}(\sigma)$$

by (2.5) and (2.1), it is sufficient to show that $MC_i^{\left(v_C^{\rho}\right)^{L|_C}}(\sigma) \geq 0$ for all $\sigma \in \Sigma(C)$ and $MC_i^{\left(v_C^{\rho}\right)^{L|_C}}(\sigma) > 0$ for some $\sigma \in \Sigma(C)$. We have

$$\begin{aligned}
& MC_i^{\left(v_C^{\rho}\right)^{L|_C}}(\sigma) \\
= \;& (v_C^{\rho})^{L|_C}(K_i(\sigma)) - (v_C^{\rho})^{L|_C}(K_i(\sigma)\backslash\{i\}) \\
\overset{(2.5)}{=} \;& \sum_{S\in\mathcal{C}\left(K_i(\sigma), L|_{K_i(\sigma)}\right)} v_C^{\rho}(S) - \sum_{S\in\mathcal{C}\left(K_i(\sigma)\backslash\{i\}, L|_{K_i(\sigma)\backslash\{i\}}\right)} v_C^{\rho}(S) \\
\overset{(4.3)}{=} \;& \sum_{S\in\mathcal{C}\left(K_i(\sigma), L|_{K_i(\sigma)}\right)} v\left(K_C^{-}(\rho)\cup S\right) - v\left(K_C^{-}(\rho)\right) \\
& \quad - \sum_{S\in\mathcal{C}\left(K_i(\sigma)\backslash\{i\}, L|_{K_i(\sigma)\backslash\{i\}}\right)} \left(v\left(K_C^{-}(\rho)\cup S\right) - v\left(K_C^{-}(\rho)\right)\right) \\
= \;& v\left(K_C^{-}(\rho)\cup C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)\backslash\{i\}\right) - v\left(K_C^{-}(\rho)\right) \\
& - \sum_{S\in\mathcal{C}\left(C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)\backslash\{i\}, L|_{C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)\backslash\{i\}}\right)} \left(v\left(K_C^{-}(\rho)\cup S\right) - v\left(K_C^{-}(\rho)\right)\right)
\end{aligned}$$

where the last equation employs the fact that i is a Null player and that removing i just splits $C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)$. Set

$$\mathcal{C}\left(C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)\backslash\{i\}, L|_{C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)\backslash\{i\}}\right) = \{S_1, \ldots, S_m\}.$$

Then, it is easy to see that $MC_i^{\left(v_C^{\rho}\right)^{L|_C}}(\sigma) = 0$ for $m = 1$, i.e., if removing i does not split $C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)$. Since $|\mathcal{C}(N,L)| < \left|\mathcal{C}\left(N\backslash\{i\}, L|_{N\backslash\{i\}}\right)\right|$, removing i splits

$C_i\left(K_i(\sigma), L|_{K_i(\sigma)}\right)$ for some $\sigma \in \Sigma(C)$. Setting $Q := K_C^-(\rho)$, one obtains

$$MC_i^{\left(v_C^\rho\right)^{L|_C}}(\sigma) = v\left(Q \cup \bigcup_{k=1}^m S_k\right) - v(Q) - \sum_{k=1}^m \left(v(Q \cup S_k) - v(Q)\right).$$

Successive application of the strict convexity of $\left(N\backslash\{i\}, v|_{N\backslash\{i\}}\right)$ finally implies $MC_i^{\left(v_C^\rho\right)^{L|_C}}(\sigma) > 0$, which proves the claim. ∎

Corollary 6.1. *If i is a Null player in (N, v) and if for all $j, k \in N\backslash\{i\}$, $j \neq k$ such that $ij, ik \in L$ we have $jk \in L$ then* $\mathrm{Ow}_i^\#(N, v, L) = 0$.

The hypothesis of this corollary implies that removing the Null player i never splits a component containing i, even if one restricts attention to subsets of N. This means that i is superfluous with respect to connecting players. Since i also is unproductive, player i should obtain a zero payoff.

Proof. The proof is analogous to the proof of Theorem 6.1. In particular, one employs the fact that one always has $m = 1$ under the hypothesis of the corollary. ∎

7. Conclusion

Myerson (1980) extends his value to the class of TU games with a conference structure (hypergraph on the player set) (henceforth CF-games and CF-value) which we call the Myerson CF-value. A hypergraph is a pair (N, H) consisting of a non-empty and finite set $N \subseteq \mathcal{U}$ and a set $H \subseteq 2^N$ the elements h of which are called hyperlinks or conference structures. Let $\mathcal{C}(N, H)$ denote the set of connected components of (N, H) and let $C_i(N, H)$ denote the component that hosts player i. Since the characterization of the Myerson CF-value is analogous to that of the Myerson value, the idea of the $\mathrm{Ow}^\#$-value can be extended to CF-games.

Slightly adapting the arguments from this paper and Myerson (1980), it is hardly more than a finger exercise to extend our CO-value into a CF-value with analogous properties: Within the definition of $\mathrm{Ow}^\#$, i.e., in (4.2) and (4.3), the graph has to be replaced by a hypergraph, and in (4.2), the Myerson value has to be replaced by the Myerson CF-value. The characterization then involves extensions of **CE**, **Q**, **CF**, and **CM**. Those of **CE** and **CM** are natural. The obvious extension of **Q** would require $\varphi\left(N, v, 2^N\right) = \varphi(N, v, \emptyset)$, but in view of the definition of the Myerson CF-value, 2^N can be replaced by L^N. Besides **CE**, the Myerson CF-value is characterized by the following modification of **F**: For all $h \in H$ and $i, j \in h$, we have

$$\varphi_i(N,v,H) - \varphi_i(N,v,H\backslash\{h\}) = \varphi_j(N,v,H) - \varphi_j(N,v,H\backslash\{h\}).$$

This translates into the following extension of **MF**: For all $h \in H$ and $i, j \in h$,

$$\begin{aligned}\varphi_i(N,v,H) &- \varphi_i\left(N_i(H,h), v|_{N_i(H,h)}, H\backslash\{h\}|_{N_i(H,h)}\right)\\ &= \varphi_j(N,v,H) - \varphi_j\left(N_j(H,h), v|_{N_j(H,h)}, H\backslash\{h\}|_{N_j(H,h)}\right)\end{aligned}$$

where

$$N_i(H,h) := N\backslash\left(C_i(N,H)\backslash C_i(N,H\backslash\{h\})\right).$$

It is easy to see that for hypergraphs containing just two-player hyperlinks the modified axioms become the original ones.

References

Bilbao, J. M., Jiménez, N. and López, J. J. (2006). A note on a value with incomplete information, *Games and Economic Behavior* **54**(2): 419–429.

Borm, P., Owen, G. and Tijs, S. (1992). On the position value for communication situations, *SIAM Journal on Discrete Mathematics* **5**: 305–320.

Hamiache, G. (1999). A value with incomplete information, *Games and Economic Behavior* **26**: 59–78.

Hart, S. and Kurz, M. (1983). Endogenous formation of coalitions, *Econometrica* **51**: 1047–1064.

Hart, S. and Kurz, M. (1984). Stable coalition structures, *in* M. J. Holler (ed.), *Coalitions and Collective Action*, Vol. 51, Physica-Verlag, Wuerzburg/Vienna, pp. 235–258.

Khmelnitskaya, A. B. and Yanovskaya, E. B. (2007). Owen coalitional value without additivity axiom, *Mathematical Methods of Operations Research* **66**(2): 255–261.

Meessen, R. (1988). *Communication games*, Master's thesis, Department of Mathematics, University of Nijmegen, the Netherlands. (in Dutch).

Myerson, R. B. (1977). Graphs and cooperation in games, *Mathematics of Operations Research* **2**: 225–229.

Myerson, R. B. (1980). Conference structures and fair allocation rules, *International Journal of Game Theory* **9**: 169–182.

Owen, G. (1977). Values of games with a priori unions, *in* R. Henn and O. Moeschlin (eds), *Essays in Mathematical Economics & Game Theory*, Springer, Berlin et al., pp. 76–88.

Shapley, L. S. (1953). A value for n-person games, *in* H. Kuhn and A. Tucker (eds), *Contributions to the Theory of Games*, Vol. II, Princeton University Press, Princeton, pp. 307–317.

van den Nouweland, A. (1993). *Games and Graphs in Economic Situations*, PhD thesis, Tilburg University, The Netherlands.

Vázquez-Brage, M., García-Jurado, I. and Carreras, F. (1996). The Owen value applied to games with graph-restricted communication, *Games and Economic Behavior* **12**: 42–53.