COORDINATION, SALIENCE, AND SYMMETRIES IN FRAMED STANDARD FORMS

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Abstract. Although there is some common understanding among game theorists that focal points guide human behavior, only recently have attempts been made to formalize the underlying idea of salience in static contexts. Bacharach (1991, 1993), Janssen (1995), and Sugden (1995) formalize the players' descriptions of the strategies and, in many cases, derive focal points and salience in the same way as intuition does. However, these approaches apply only to special classes of games and leave the use of player descriptions and some subtle kinds of salience unexplained. This paper proposes a general notion of frames for standard forms (Harsanyi & Selten 1988) as a formal structure that represents the players' non-strategic descriptions of the game via multidimensional labels of strategies. Resting upon this notion, isomorphisms of framed standard forms (FSFs) and symmetry invariance are introduced. It is shown that the other approaches can be represented by FSFs. *Journal of Economic Literature* Classification Number: C72.

1. Introduction

Traditional game theoretic solution concepts—at least implicitly—regard the strategy and player labels used as the game theorist's ones. Thus, rational players should not be concerned with these labels. Harsanyi & Selten (1988, pp. 70) express this view most clearly in their requirement of invariance with respect to isomorphisms for solutions of games.

Consider, to take the most striking example, two-player one-shot matching games—both players have the same (finite) strategy set and the same payoff function such that if they choose the same strategy, each of their payoffs is 1 and otherwise it is 0. Since the players cannot distinguish their strategies in terms of payoff alone, symmetry invariance forces them to randomize over their strategy sets. This prescription, however, is highly inefficient—it renders the players with payoffs far lower than in the case of coordination.

Observations in real life (e.g., in natural meeting problems) show that people do not blindly randomize over the whole range of the possible options. On the contrary, they are often quite successful in achieving coordination—at least much more successful than one could predict from randomizing

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guided by the symmetry invariance requirement. In real life, of course, people do not perceive their decision problems only in terms of payoff differences between strategies. Besides the payoff structure, other features of the game are systematically exploited by the players in their decision process. It matters how a player describes the game to himself and how he expects the other players to do so. Besides Schelling's early sample, recent experiments (e.g., Mehta, Starmer & Sugden 1994a, Mehta, Starmer & Sugden 1994b, Bacharach & Bernasconi 1997) support these observations.

Schelling (1960) introduced the notion of *focal points* to denote this systematic use of the players' strategy labels in strategic contexts, especially in coordination games. He suggests that "[m]ost situations ... provide some clue for coordinating behavior, some focal point for each person's expectation of what the other expects him to expect to be expected to do." These clues are vaguely characterized as commonly apprehended prominent or conspicuous, and unique. Similarly, Lewis (1969, p. 35) coined the term of *salience*—uniqueness in some conspicuous respect.

Although there is some common understanding among game theorists that focal points guide human behavior, only recently have attempts been made to formalize the underlying idea of salience (Bacharach 1991, Bacharach 1993, Sugden 1995, Janssen 1995). The reason for this seems to be partly that the notion of salience is intrinsically difficult to formalize. Schelling (1960, p. 58) suggests that "... we are dealing with imagination as much as with logic; and the logic is of a fairly casuistic kind. Poets may do better than logicians at this game, which is more like 'puns and anagrams' than like chess. Logic helps ... but usually not until imagination has selected some clue to work on from among the concrete details of the situation." Clearly, formalizations of salience will have to catch some of this imagination. Moreover, they involve two parts—(a) some formal structure that represents the players' apprehension of the game situation and (b) a mechanism to derive a salient option from this structure. As Goyal & Janssen (1996, p. 45) remark, this kind of formalization of salience is to be distinguished from a rationalization of choosing the salient (e.g., Gauthier 1975).

This paper is organized in the following way. The second section discusses formalizations of focal points and salience that are based on exogenous descriptions. Basic notation and definitions of standard forms are given in the third section. The next section introduces the concepts of framed standard forms (FSFs) and isomorphisms of FSFs. In the fifth section, the proposed concept is applied to various examples, and its relation to other concepts is discussed. The final section gives some concluding remarks.

2. Formalizations of focal points and salience

This paper mainly considers the formalization of exogenous descriptions of games and the derivation of focal points based on it. A static perspective is adopted—it is not explained how descriptions arise. The analysis of static focal points is interesting on the one hand because people do in fact frequently face situations where the descriptions are given. In dynamic games, the description of a game may be the result of the players' previous actions or, more general, the history of the game, nevertheless, at a given time, this description is given to the players. In this sense, one could say that static focal points are the building blocks of dynamic focal points. What the dynamic perspective adds is that the players may anticipate how present actions influence their future descriptions of the game, and thereby influence their future actions and the resulting payoffs. In turn, this foresight may lead the players to take actions that produce a favorable description (Crawford & Haller 1990, p. 591). In the following, some formalizations of static focal points are presented and discussed.

2.1. **Rationalizing the choice of the salient.** Gauthier (1975) introduces an approach that aims at a rationalization of choosing salient options, and in doing so he clarifies the role of salience. He suggests that the decision process of players involved in a matching game takes two steps:

1. The players restrict the options which they consider by reconceiving their option sets.
2. The players choose their options that result in a strictly payoff dominant equilibrium.

In order to reconceive their strategy space, first the players have to conceive one of the strategies as the commonly salient one. Next, the players decide to restrict the set of possible actions to just two actions—either to choose the salient option or to ignore salience and to randomize over all options. Finally, both players' choosing the salient option is the only payoff dominant equilibrium.

Although the payoff dominance requirement is not uncontroversial from a strictly individualistic standpoint, this two-step process seems to be an important step towards a rationalization of salience (Bacharach 1991), even in view of Gilbert's (1989) criticism. Clearly, for this purpose other ways of restricting the perceived-option set have to be used.

2.2. **Variable frame theory of focal points.** Bacharach's (1991, 1993) variable frame theory (VFT) extends a class of base games (two-player games with the identical option set S and the identical payoff function u) to a special kind of Bayesian game—the variable universe game (VUG). The players' types in these Bayesian games differ in the way they apprehend the game, and in particular the option set. The different apprehensions are represented by different *repertoires* r, that is, collections of *families* F from a family set $\mathbf{F}$. Families are non-intersecting sets of related concepts that can be used by players to describe certain properties attributable to the players' options, for example, the `color` family comprising color concepts such as `red`, `green` and `blue` or the `shape` family of the shape concepts such as `round`, `triangular` or `square`.

The repertoires may give rise to different subjectively perceived option sets. These sets are determined by a *coverage* function A such that $A(r)$ denotes the perceived-option set for repertoire r. Bacharach & Bernasconi (1997) specify the coverage function A in the following way. Each concept in one of the families of a repertoire r that actually is attributed to one of the options contributes one pure perceived strategy to the set $A(r)$, namely, the randomization over all pure strategies that are characterized by this concept. In addition, $A(r)$ contains the randomization over the whole option set.

Although the repertoires are likely to vary, the players may have an idea of how they are distributed in the relevant population of players or may have some (subjective) expectation with respect to the repertoires of the other players. Common knowledge of a shared probability distribution can be interpreted as a formal expression of the players' awareness of a common culture in a population. Some special assumptions about the distribution of the repertoires are made. Let $V(F)$ denote the common probability of the family F. These probabilities can be viewed as the shares of those players in the population that realize that family. Assuming that the families are independently distributed, the share of repertoire r is given by

$$V\left(r\right)=\prod_{F\in r}V\left(F\right)\prod_{F\in\mathbf{F}\backslash r}\left(1-V\left(F\right)\right).$$

The common conditional probability that a player with the repertoire r expects the other player to have the repertoire $r^{'}$ is defined as

$$V(r^{'}|r)=\begin{cases}\prod_{F\in r'}V\left(F\right)\prod_{F\in r\backslash r'}\left(1-V\left(F\right)\right) & : \ r^{'}\subseteq r\\ 0 & : \ r^{'}\nsubseteq r\end{cases}. \tag{1}$$

This means that a player cannot think of families that are not contained in his repertoire. He regards repertoires that are not subrepertoires of his own repertoire as impossible. Note, this kind of conditional probability cannot be derived from a common prior on pairs of types by the Bayesian rule. The players' strategies ρ_i are mappings from the repertoire set $\mathbf{R}=2^{\mathbf{F}}$ to the set of mixed options Σ such that $\rho_i(r)\in A(r)$. The expected payoff of player i with the repertoire r is given by

$$u_i(\rho|r)=\sum_{r^{'}\in\mathbf{R}}V(r^{'}|r)u(\rho_i(r)\rho_{-i}(r^{'}))=\sum_{r^{'}\subseteq r}V(r^{'}|r)u(\rho_i(r)\rho_{-i}(r^{'})).$$

Bacharach introduces symmetric Bayesian equilibria that are not payoff dominated as the solutions of variable universe games.

Example 2.1 (Blockmarking)**.** *Blockmarking is a two-player matching game in whitch both players independently choose among children's wooden play-blocks on a tray. Three sets of blocks are considered:*

1. *Five identical blocks,*
2. *Five identical blocks that are identical except of that one block is red and the rest are yellow,*

3. *Twenty identical blocks that are identical except of that two are red and 18 are yellow, and, not so obvious, one of the yellow blocks has a wavy grain of wood whereas the rest have a straight one.*

Bacharach analyzes the three games of Blockmarking in the VFT framework. In Blockmarking 1, intuition says to randomize over all five blocks and so does VFT. Taking the red block is the obvious way to play Blockmarking 2, and this also is prescribed by VFT. In Blockmarking 3, the solution depends on how probable it is that a player realizes the differences in the grain of wood. Picking a red block is prescribed if this probability is less than $\frac{1}{2}$, and taking the wavy yellow block is prescribed if it is greater than $\frac{1}{2}$. Again, this is in line with intuition—the likelier it is that the other player realizes the grain pattern the likelier it is for a player that realizes grain patterns to take yellow block with the wavy grain of wood.

The assumptions for the conditional probabilities $V(r^{'}|r)$ and the coverage function A are rather special but could easily be relaxed without essentially changing the approach. The above coverage function does not account for the possibility that the players combine the information of several families. As it is suggested by Choose an Object 1, however, this is not always a disadvantage. Alternatively, the set $A(r)$ could contain the randomizations over all groups of strategies that are indistinguishable with respect to the repertoire r.

Dropping the independence assumption, the probabilities $V(r)$ could be introduced as primitives, and the conditional probabilities $V(r^{'}|r)$ be derived as

$$V(r^{'}|r) = \begin{cases} \frac{V(r^{'})}{\sum_{r^{''}\subseteq r} V(r^{''})} & : \; r^{'} \subseteq r \\ 0 & : \; r^{'} \nsubseteq r \end{cases} .$$

The assumption that $V(r^{'}|r) = 0$ if $r^{'} \nsubseteq r$ seems to be quite sensible in a wide range of situations—situations in which the strategies and their descriptions are not known to the players in advance. In this case, a sophisticated player could infer anything about the choices of players that realize families outside of his repertoire (Bacharach 1993, p. 269).

Example 2.2 (Useful Sophistication)**.** *In this matching game the objects are shown to the players in advance: four red cubes, two white balls, and three green balls. Both players know that they have to make their choice with some probability v in absolute darkness.*

In Useful Sophistication, however, a player who is to decide in darkness (darkness type) knows that a player who is to decide in brightness (brightness type) can distinguish two of the balls from the others. He could infer that a brightness type is going to pick a white ball, and that he should therefore pick a ball. For $v > \frac{5}{6}$, besides both types' picking a cube, picking a white ball (brightness type) or picking a ball (darkness type) is a Pareto optimal Bayesian equilibrium where the brightness type receives a higher payoff than in the other Pareto optimal Bayesian equilibrium—sophistication is useful. This equilibrium, however, cannot be supported within Bacharach's specification.

Example 2.3 (Take Two Black Balls)**.** *Three players—two women and a man—are each given a basket containing one white ball and one black ball. In separate rooms and without the possibility of communication, the players take one ball from their basket and hand it to the game master. If exactly two black balls have been chosen, all players get a prize. Otherwise they get nothing.*

In Take Two Black Balls, both strategies of the three players are distinguished by the payoff structure, but the players themselves are not. However, the players' problem is to find out two of them to take the black ball. Intuitively it is clear that both women should choose the black ball. Since VUGs do not model player attributes and consider only two players, again, this intuitive solution cannot be derived within the VUG framework.

2.3. **Iterative reduction of the option set.** Janssen (1995) introduces a solution concept for a specification of VUGs in which the assignment of strategies to the families' concepts is explicitly modelled as their membership in cells of *attributes*—partitions of the strategy set. These attributes represent the families, and sets of attributes represent the repertoires of VFT. Core of the solution concept is a notion of symmetry between groups of the players' strategies and an iterative reduction mechanism for the sets of groups formed by the attribute sets. After this reduction, payoff dominance determines the solution. For the class of VUGs based on matching games Janssen claims generic uniqueness, equilibrium, and efficiency properties for this solution concept. Partly, however, these results do not hold generically (Casajus 1997).

Example 2.4 (Choose an Object)**.** *Choose an Object is a matching game with three different sets of objects:*

1. *(1) white ball, (2) white cube, (3) red ball, (4) yellow cube, (5) yellow ball, (6) yellow ball;*
2. *(1) white ball, (2) green ball, (3) red ball, (4) yellow cube, (5) yellow ball, (6) yellow ball;*
3. *(1) white ball, (2) white cube, and (3) red ball.*

Janssen demonstrates his solution with a number of matching games. The red ball is selected in Choose an Object 1. Intuitively, this result is obtained by considering different framings of the option set. Viewing the objects with respect to their shape alone, none of the objects is unique, but viewing them with respect to their color, the red ball is unique and should therefore be chosen. In Choose an Object 2, several of the objects are unique with respect to shape or color. However, the three balls are somehow symmetric with respect to color and should be assigned the same probability. Then, taking the yellow cube is the payoff dominant choice. If the perceptions of shape and color are not certain and equally probable, this is also Janssen's solution.

In the case of Choose an Object 3, things are a bit more complicated. Both the white cube and the red ball are unique with respect to one attribute and should be treated in the same way. A similar argument can be applied to the groups of the white and of the round objects; the members of both groups—all objects—should be given the same probability. Payoff dominance, then, leads to taking either the white cube or the red ball, each with probability $\frac{1}{2}$. But, as Janssen points out, the players may be able to recognize a more subtle kind of salience in this game. They may realize that both the white cube and the red ball are are unique with respect to one attribute, whereas the white cube is not unique with respect to any of the attributes. Thus, the white cube is salient and should be chosen. In order to derive this result, Janssen has to add an additional attribute that captures this uniqueness of the white cube. One could say that the salience of the white cube is put in exogenously. An endogenization of this kind of salience seems to be desirable.

2.4. **Label procedures.** While Bacharach and Janssen invoke the principle of insufficient reason to enforce randomization over 'similar' strategies, Sugden (1995) considers a game structure that may prevent the players from selecting a single strategy from a set of 'similar' strategies. In fact, he introduces the labelling (label procedure) of two-person games as probability distributions over pairs of label functions. Label functions assign different labels to each of a player's strategies. As the players can identify the strategies by their labels only, and not the strategies themselves, this kind of labelling may reduce the strategy sets that the players consider. Thus, Sugden's approach accounts for a precise definition of the players' actual strategic possibilities with respect to their perception of the game.

This is easily demonstrated with one of Sugden's 'Choose a Disc' examples. Two players are each given a bag with the same set of discs that are marked by letters that are invisible to the players. The discs are identical except that one of the discs is green and all other discs are red. Sitting in separate rooms, their task is to take out the discs one by one, place them on the table, and then choose one of them. The players win a prize if they choose the same disc. Suppose that the players are completely color-blind. All discs are identical in appearance to the players, and the only means of distinguishing them is the order of taking them out. An English speaking player would label the discs as `first`, `second`, `third`, `fourth` and `fifth`. In this setting, it is plausible to assume that both orders are statistically independent and that all pairs of orders are equally likely. By choosing the strategy labelled `second`, a player can expect to have taken any of his strategies with the same probability. Suppose now that the players have normal vision. It is likely that the players will label the discs as `green`, `first_red`, second_red, third_red, and fourth_red. In line with the above argument, a player will always expect to have taken a strategy that assigns the same probability to any of his choices labelled red. This, given the payoff dominance criterion, rationalizes the choice of the green disc.

Consider a setting in which the discs are taken out publicly. Then, the players' orders of numbering are no longer statistically independent, and each of the red discs is labelled in the same way for both players. As in the case where there is no uncertainty about the strategies' labels, the strategy set is not restricted by the labelling. Another limitation of this approach is the fact that the labels are onedimensional only. In contrast to Bacharach and Janssen the labels themselves have no structure. Thus, solutions as in Choose an Object are not accessible.

3. Standard forms and solutions

Point of departure for the suggested framing of games is the standard form introduced by Harsanyi & Selten (1988), henceforth HS. In this section, first, some basic notation, definitions, and properties of standard forms are introduced, and then some requirements on the solutions.

3.1. **Standard forms.** The player set N is a non-empty finite set of integers. Player i's agent set M_i is a non-empty finite set of agents ij where the first integer represents the player to whom the agent belongs and the second integer represents the agent. The union of all M_i is denoted by M. Each agent ij has a non-empty finite choice set Φ_{ij} of choices ϕ_{ij}. A pure strategy ϕ_i of player i is a collection of his agents' choices $(\phi_{ij})_{M_i}$. Player i's pure-strategy set is $\Phi_i := \bigtimes_{ij\in M_i}\Phi_{ij}$. A pure-strategy combination ϕ is a collection of the players' pure strategies $(\phi_i)_N$ or a collection of the agents' choices $(\phi_{ij})_M$. The pure-strategy combination set is $\Phi := \bigtimes_{i\in N}\Phi_i = \bigtimes_{ij\in M}\Phi_{ij}$. The payoff function H assigns a payoff vector $H(\phi) = (H_i(\phi))_N$ to each $\phi \in \Phi$. Standard forms are pairs (Φ, H). G, G' and G'' denote the standard forms (Φ, H), (Φ', H') and (Φ'', H'').

$G_m[n]$ stands for the standard form of a two-player matching game with n strategies for both players. Since the players have only one agent, the agents' index is dropped. We set $N_m = \{1, 2\}$, for both players i $\Phi_i^m = \{\phi_i^1, \phi_i^2, \dots, \phi_i^n\}$, and $H_i^m(\phi_1^j, \phi_2^k) = 1$ if $j = k$, and $H_i^m(\phi_1^j, \phi_2^k) = 0$ if $j \neq k$.

A local strategy b_{ij} of an agent ij is a probability distribution over his choice set Φ_{ij}. The probability assigned to ϕ_{ij} by ij is denoted by $b_{ij}(\phi_{ij})$. The set of all local strategies of ij is denoted B_{ij}. A behavior strategy b_i for player i is a collection $(b_{ij})_{M_i}$ of his agents' local strategies. The set of all behavior strategies of player i is denoted by B_i. A behavior-strategy combination is a collection $(b_i)_N$ or $(b_{ij})_M$. The set of all behavior-strategy combinations is denoted by B. The payoff function H is extended to B in the usual way. For each non-empty set of agents $C \subseteq M$, a collection $b_C = (b_{ij})_C$ of local strategies $b_{ij} \in B_{ij}$ is called a subcombination for C. The set of all subcombinations for C is denoted by B_C. Sometimes the set

$M \backslash C$ is abbreviated by $-C$ and the set $M \backslash M_i$ is abbreviated by $-i$. The centroid of an agent's choices is his strategy that uses all choices with the same probability. $c(C)$ denotes the subcombination b^C that consists of the centroids of the agents in C.

A behavior-strategy combination $b \in B$ is a local equilibrium if $H_i(b_{ij}b_{-ij}) = \max_{b'_{ij} \in B_{ij}} H_i(b'_{ij}b_{-ij})$ holds for each agent $ij \in M$. For the class of interior substructures of standard forms with perfect recall, HS show that local equilibria are also equilibria. A solution function L for a class of standard forms assigns an equilibrium to every standard form of this class.

3.2. **Symmetry invariance.** HS (pp. 70) give the idea that the players cannot base their decisions on the game theorist's labels a most general formulation. For this purpose, they first introduce the notion of isomorphisms of standard forms, and then derive an invariance property on the solution of standard form with respect to these isomorphisms.

Definition 3.1. *An isomorphism from $G = (\Phi, H)$ to $G' = (\Phi', H')$ is a one-to-one mapping f from Φ onto Φ' such that the following conditions are satisfied:*

1. *There is a one-to-one mapping σ from N onto N', for every player $i \in N$ a one-to-one mapping ρ_i from M_i onto $M'_{\sigma(i)}$, and for every agent $ij \in M$ a one-to-one mapping f_{ij} from Φ_{ij} onto $\Phi'_{\sigma(i)\rho(j)}$ such that for all agents $ij \in M$ and all pure-strategy combinations $\phi \in \Phi$ holds*
$$f(\phi)_{\sigma(i)\rho_i(j)}(\phi) = f_{ij}(\phi_{ij}).$$
2. *For all players $i \in N$, there are constants $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that for all pure-strategy combinations $s \in S$ holds*
$$H'_{\sigma(i)}(f(\phi)) = \alpha_i H_i(\phi) + \beta_i.$$

An isomorphism involves renamings of players (σ), agents $(\rho_i)_N$, and choices $(f_{ij})_M$ and separate positive affine transformations of each players' payoff functions. The mapping f is defined by the first condition via the mappings σ, $(\rho_i)_N$ and $(f_{ij})_M$. In the usual way, the mapping f is extended to domain B by $f(b)_{\sigma(i)\rho_i(j)}(f_{ij}(\phi_{ij})) := b_{ij}(\phi_{ij})$ for all $i \in N$ for all $\phi_{ij} \in \Phi_{ij}$ and all $b \in B$. Thus, the invariance property says that the solution should neither be influenced by renamings of the players, agents, and choices nor be influenced by positive affine transformations of the payoffs. Since all mappings involved are bijective, and compositions and inverses of positive affine transformations are positive affine, the inverse and the composition of isomorphisms is also an isomorphism.

Definition 3.2. *A solution function L for a class of standard forms is called invariant with respect to isomorphisms if for every isomorphism f from G to G' we have $f(L(G)) = L(G')$.*

The isomorphisms from a standard form to itself are called symmetries. A behavior-strategy combination $b \in B$ in G is symmetry invariant if for all symmetries f of G $f(b) = b$ holds. Invariance with respect to isomorphisms requires the solution of standard forms to be symmetry invariant. This requirement does not go too far. Nash's (1951, p. 289) theorem assures that symmetry invariant local equilibria exist for every standard form.

3.3. **Cell and truncation consistency.** To identify subgames in the standard form representation of extensive forms, HS (pp. 90) introduce the notion of cells. To be precise, HS show that subgames in extensive forms with perfect recall always correspond to cells of interior substructures of their standard form representation. However, not every cell corresponds with a subgame of the underlying extensive form. Generally speaking, cells are strategically independent parts of standard forms.

Let C be a non-empty proper subset of M. Let $G^C = (\Phi_C, H^C)$ be the standard form that results from G by fixing the players in $M \backslash C$ at their centroids. That is, for all $ij \in C$ and all $\phi_C \in \Phi_C$ we have $H_i^C(\phi_C) = H_i(\phi_C, c(-C))$.

Definition 3.3. *The set C forms a cell G^C of G if for every $\phi_{-C} \in \Phi_{-C}$ and every i with $M_i \cap C = \emptyset$, a number $\alpha_i(\phi_{-C}) > 0$ and a number $\beta_i(\phi_{-C})$ can be found such that*

$$H_i(\phi_C \phi_{-C}) = \alpha_i(\phi_{-C}) H_i^C(s^C) + \beta_i(\phi_{-C}).$$

A cell is called elementary if it does not contain another cell as a proper subset. HS show that elementary cells do not intersect.

The truncation of a standard form G with respect to a cell G^C and a solution function L is the game $\bar{G} = (\Phi_{-C}, H^{-C})$ that results from G by fixing the agents in C at their local strategies in the solution $L(G^C)$ of G^C. That is, for all $ij \in -C$ and all $\phi_{-C} \in \Phi_{-C}$, we have $H_i^{-C}(\phi_{-C}) = H_i(L(G^C), \phi_{-C})$. The main truncation of a standard form with respect to a solution function L is the game that results by fixing agents of the elementary cells at their solutions. A game is called indecomposable if it does not contain cells.

The idea that the solution of a strategically independent part of a game should not be influenced by the outside part is expressed with the requirement of cell consistency and the complementary requirement of truncation consistency.

Definition 3.4. *A solution function L for a class of standard forms is cell consistent if for each cell G^C of a standard form G, $L(G)$ prescribes the strategies for the agents in C as $L(G^C)$.*

Definition 3.5. *A solution function L for a class of standard forms is truncation consistent if for each cell G^C of a standard form G, $L(G)$ prescribes the strategies for the agents in $-C$ as $L(\bar{G}, \mathbb{F}^{-C})$ where $\bar{G}$ is the truncation of G with respect to C and L.*

According to the *composition lemma* (HS, p. 102), the combination of the solutions of the elementary cells and main truncation is also an equilibrium of the whole game. The *extension lemma* (HS, p. 103) assures that a solution function on the set of indecomposable games can be uniquely extended to a solution function on the set of all standard forms that is cell and truncation consistent. Furthermore, according to the *lemma on invariance with respect to isomorphisms* (HS, p. 123), for a solution function for indecomposable games that is invariant with respect to isomorphisms, the extension is also invariant with respect to isomorphisms.

4. Framed standard forms

4.1. **Frames.** Symmetry invariance is a very strong requirement on solutions for standard forms. This requirement, however, correctly reflects the fact that players cannot make any use of labels. This is justified because the strategy and player labels in games are the game theorist's one's. In standard forms, as well as in normal and extensive forms, the labels assigned by the players to choices, agents or players are usually not formalized. Thus, in some games there are a lot of symmetries that restrict the set of symmetry invariant strategies. In the following, frames of standard forms are introduced. These frames add information about the players descriptions of the game situation to the standard form via a multidimensional labelling of the choices.

The frame $\mathbb{F}$ of a standard form $G = (\Phi, H)$ is a tuple $(\Lambda, \mathfrak{A}, (\ell_{ij\mathfrak{a}})_{M\times\mathfrak{A}})$, where Λ denotes the non-empty finite set $\{\lambda, \lambda^{'}, \ldots, \lambda^{'\cdots'}\}$ of choice labels, $\mathfrak{A}$ the non-empty finite set of attributes $\{\mathfrak{a}, \mathfrak{a}^{'}, \ldots, \mathfrak{a}^{'\cdots'}\}$, and $\ell_{ij\mathfrak{a}}$ the label function $\Phi_{ij} \longmapsto \Lambda$ for every agent $ij \in M$ and every attribute $\mathfrak{a} \in \mathfrak{A}$. The set $\mathbb{L} := \bigtimes_{k\in M}\Lambda$ comprises all available denotations (label tuples) of the choices in G. A pair $(G, \mathbb{F})$ is called framed standard form (FSF). $\mathbb{F}$, $\mathbb{F}^{'}$ and $\mathbb{F}^{''}$ denominate the frames $(\Lambda, \mathfrak{A}, (\ell_{ij\mathfrak{a}})_{M\times\mathfrak{A}})$, $(\Lambda^{'}, \mathfrak{A}^{'}, (\ell^{'}_{ij\mathfrak{a}})_{M^{'}\times\mathfrak{A}^{'}})$ and $(\Lambda^{''}, \mathfrak{A}^{''}, (\ell^{''}_{ij\mathfrak{a}})_{M^{''}\times\mathfrak{A}^{''}})$ of the standard forms G, $G^{'}$ and $G^{''}$.

Frames consist of two parts—the language part $(\Lambda, \mathfrak{A})$ and the labelling part $(\ell_{ij\mathfrak{a}})_{M\times\mathfrak{A}}$. While the language part is independent of the framed game, the labelling is related both to the game and to the language. Considering $(\Lambda, \mathfrak{A})$ as a language, one can think of a choice's denotation as a sentence that describes it. $\mathbb{L}$ contains all sentences that can be formed in this language. The labels λ stand for the words, the attributes $\mathfrak{a}$ for the individual parts of a sentence and the whole attribute set $\mathfrak{A}$ for the common structure of these sentences. In natural languages, of course, not all words will fit in every position of a sentence. But, as it is shown, there are applications of framings in which this wider definition proves useful (see section 5.8).

4.2. **Isomorphisms.** In many contexts, it seems to be quite plausible to assume that the solution of an FSF does not depend on the language used by the players to denote choices, agents and players. Put differently, denotations have no intrinsic meaning to the players. Thus, the translation of these denotations into another language should not change the solutions. This requirement is formally expressed with the following definitions:

Definition 4.1 (Isomorphisms of FSFs)**.** *An isomorphism from a FSF* $(G, \mathbb{F})$ *to* $(G', \mathbb{F}')$ *is a pair of one-to-one mappings* (f, ξ), f *from* Φ *onto* Φ' *and* ξ *from* $\mathbb{L}$ *onto* $\mathbb{L}'$, *such that the following conditions are satisfied:*

1. *The mapping* f *is an isomorphism from* G *to* G'.
2. *There are one-to-one mappings* $\xi_{\mathfrak{A}}$ *from* $\mathfrak{A}$ *onto* $\mathfrak{A}'$ *and* ξ_Λ *from* Λ *onto* Λ' *such that for all* $\mathfrak{a} \in \mathfrak{A}$ *and all* $\boldsymbol{\lambda} \in \mathbb{L}$ *holds*
$$\xi(\boldsymbol{\lambda})_{\xi_{\mathfrak{A}}(\mathfrak{a})} = \xi_\Lambda(\boldsymbol{\lambda}_{\mathfrak{a}}).$$
3. *For every agent* $ij \in M$ *for every choice* $\phi_{ij} \in \Phi_{ij}$ *and for every attribute* $\mathfrak{a} \in \mathfrak{A}$ *holds*
$$\xi_\Lambda\left(\ell_{ij\mathfrak{a}}\left(\phi_{ij}\right)\right) = \ell'_{\sigma(i)\rho(j)\xi_{\mathfrak{A}}(\mathfrak{a})}\left(f_{ij}\left(\phi_{ij}\right)\right).$$

Definition 4.2. *A solution function* L *for a class of FSFs is invariant with respect to isomorphisms if for every isomorphism* (f, ξ) *from* $(G, \mathbb{F})$ *to* $(G', \mathbb{F}')$ *we have* $f(L(G, \mathbb{F})) = L(G', \mathbb{F}')$.

The first condition in definition 4.1 secures that the players are able to distinguish their strategies in terms of payoffs but do not have access to the game theorist's labels. According to the second condition, the involved translation of the sentences (label tuples) from $\mathbb{L}$ into $\mathbb{L}'$ can be done by translating the words (labels) and the parts of sentence (attributes) separately. Finally, the third condition requires that the relation between the payoff structure and the structure of the players' labels is preserved by an isomorphism. This implies that the players' description of the game combines payoff structure and label structure. Since the involved translation of the language is bijective, the composition and the inverse of isomorphisms of FSFs are also isomorphisms.

Lemma 4.1. *Let* (f, ξ) *be an isomorphism from* $(G, \mathbb{F})$ *to* $(G', \mathbb{F}')$ *and* (f', ξ') *an isomorphism from* $(G', \mathbb{F}')$ *to* $(G'', \mathbb{F}'')$. *Then, the composition* $(f' \circ f, \xi' \circ \xi)$ *is an isomorphism from* $(G, \mathbb{F})$ *to* $(G'', \mathbb{F}'')$, *and the inverse* $\left(f^{-1}, \xi^{-1}\right)$ *is an isomorphism from* $(G', \mathbb{F}')$ *to* $(G, \mathbb{F})$.

4.3. **Symmetries and symmetry invariance.** Isomorphisms from a FSF to itself are called its symmetries. Clearly, there exists a symmetry for every FSF—the identity mapping. A behavior-strategy combination $b \in B$ in a FSF $(G, \mathbb{F})$ is symmetry invariant if for each symmetry (f, ξ) of $(G, \mathbb{F})$ holds $f(b) = b$. By definition, the strategy part f of every symmetry (f, ξ) of a FSF $(G, \mathbb{F})$ is a symmetry of the standard form G. Since frames impose additional restrictions, symmetry invariance in FSFs involves a smaller set—at most the same set—of symmetries f to be considered than symmetry invariance in the underlying standard forms. Thus, the existence of symmetry invariant equilibria for standard forms implies the existence of symmetry invariant equilibria for FSFs.

Theorem 4.2. *Every FSF has a symmetry invariant equilibrium.*

Two choices (agents, players, labels, attributes) of FSF are called symmetric if there is a symmetry that maps one of them to the other. In view of lemma 4.1, this definition is equivalent to the seemingly more general definition via a chain of such symmetry mappings. Obviously, two agents (players) can be symmetric only if they have symmetric choices (agents). Together with the fact that the identity mapping is a symmetry of FSFs, lemma 4.1 implies that the symmetry of choices (agents, players, labels, attributes) is an equivalence relation.

Now, some properties of symmetries are presented that prove useful in the determination of symmetry relations in FSFs.

Lemma 4.3. *In FSFs, two choices are symmetric only if they are assigned the same number of different labels.*

Proof. Suppose the choices ϕ_{ij} and $\phi_{i'j'}$ of $(G, \mathbb{F})$ are symmetric. By definition, there is a symmetry (f, ξ) of $(G, \mathbb{F})$ such that $f_{ij}(\phi_{ij}) = \phi_{i'j'}$. Let λ be an arbitrary label and let $\mathfrak{a}$ and $\mathfrak{a}'$ be arbitrary attributes such that $\ell_{ij\mathfrak{a}}(\phi_{ij}) = \ell_{ij\mathfrak{a}'}(\phi_{ij}) = \lambda$. By definition of isomorphisms of FSFs we have $\xi_\Lambda(\ell_{ij\mathfrak{a}}(\phi_{ij})) = \ell_{\sigma(i)\rho(j)\xi_{\mathfrak{A}}(\mathfrak{a})}(f_{ij}(\phi_{ij}))$ and by the assumption $\xi_\Lambda(\lambda) = \ell_{i'j'\xi_{\mathfrak{A}}(\mathfrak{a})}(\phi_{i'j'}) = \ell_{i'j'\xi_{\mathfrak{A}}(\mathfrak{a}')}(\phi_{i'j'})$. Thus, there is bijective mapping from the set of all attributes that assign a certain label to ϕ_{ij} onto the set of all attributes that assign a certain label to $\phi_{i'j'}$. Lemma 4.1 assures that the converse is also valid. Therefore, there is bijective mapping between the label sets of both choices. Since the number of labels is finite, the cardinality of these sets must be equal. ∎

The proofs of the next two lemmas follow very much the lines of the proof of lemma 4.3 and are therefore suppressed.

Lemma 4.4. *In FSFs, two labels are symmetric only if they are assigned to the same number of choice-attribute pairs.*

Lemma 4.5. *In FSFs, two attributes are symmetric only if they are assigned the same number of different labels.*

4.4. **Cell and truncation consistency.** In order to decompose FSFs, cells and truncations of FSFs are defined in a similar way as in standard forms. A cell of a FSF $(G, \mathbb{F})$ is a FSF $(G^C, \mathbb{F}^C)$ where G^C is a cell of G and $\mathbb{F}^C = (\Lambda, \mathfrak{A}, (\ell_{ij\mathfrak{a}})_{C\times\mathfrak{A}})$ results from $\mathbb{F}$ by restricting $\mathbb{F}$ to the label functions related to the agents in C. The truncation of a FSF $(G, \mathbb{F})$ with respect to a cell $(G^C, \mathbb{F}^C)$ and a solution function L is a FSF $(\bar{G}, \mathbb{F}^{-C})$ where $\bar{G}$ is the truncation of G with respect to G^C and L, and $\mathbb{F}^{-C} = (\Lambda, \mathfrak{A}, (\ell_{ij\mathfrak{a}})_{-C\times\mathfrak{A}})$ results from $\mathbb{F}$ by restricting $\mathbb{F}$ to the label functions related to the agents in $-C$. Note, it is important that cells and truncations of FSFs contain the whole language of a frame.

A FSF is indecomposable if it does not contain cells. A cell $(G^C, \mathbb{F}^C)$ is called elementary if G^C is an elementary cell of G. The main truncation of a FSF is the truncation related to the union of the elementary cells. In accordance with the respective notions for standard forms, cell and truncation consistency for the solutions of FSFs are defined.

Definition 4.3. *A solution function L on a class of FSFs is cell consistent if for each cell $(G^C, \mathbb{F}^C)$ of a FSF $(G, \mathbb{F})$, $L(G, \mathbb{F})$ prescribes the same strategies for the agents in C as $L(G^C, \mathbb{F}^C)$.*

Definition 4.4. *A solution function L on a class of FSFs is truncation consistent if for each cell $(G^C, \mathbb{F}^C)$ of a FSF $(G, \mathbb{F})$, $L(G, \mathbb{F})$ prescribes the same strategies for the agents in C as $L(\bar{G}, \mathbb{F}^{-C})$ where $\bar{G}$ is the truncation of G with respect to C and L.*

Since cells and truncations of FSFs are defined via cells and truncations of standard forms, the combination of equilibria in a cell and the corresponding truncation is also an equilibrium of the whole FSF. It remains to show that the cell and truncation consistency requirement is not in conflict with invariance with respect to isomorphisms.

Theorem 4.6. *Let L^0 be a solution function on the class of indecomposable FSFs that is invariant with respect to isomorphisms. Its cell and truncation consistent extension on the class of FSFs is invariant with respect to isomorphisms.*

The proof of this theorem is very similar to case 3 of the proof of the *lemma on invariance with respect to isomorphisms* (HS, pp. 123). The only addition to make is to notice that an isomorphism (f, ξ) from $(G, \mathbb{F})$ to $(G', \mathbb{F}')$ carries elementary cells of $(G, \mathbb{F})$ to elementary cells of $(G', \mathbb{F}')$. As already mentioned above, the language part of a frame of an elementary cell is the same as the language part of the whole FSF. This property allows us to decompose FSFs and to apply the solution function to indecomposable FSFs only.

4.5. **Reduction of FSFs.** In some matching games, as for instance in Choose an Object 1, symmetry invariance and payoff dominance together do not produce unique results. Therefore, Janssen (1995) suggests a reduction algorithm for the strategy sets considered by the players depending on their frame (Janssen: attribute set). In the following a comparable reduction of FSFs is introduced.

Definition 4.5. *Let $(G, \mathbb{F})$ be an FSF. The reduced FSF $r(G, \mathbb{F})$ is a FSF $(G, \mathbb{F}')$ such that*

1. *The attribute set $\mathfrak{A}'$ contains only the attribute* `sym`*.*
2. *The label set Λ' is the partition of the set of all choices of G formed by the choice-symmetry relation.*
3. *The label functions assign labels to the choices contained in them; that is, $\phi^k_{ij} \in \ell_{ij\mathtt{sym}}(\phi^k_{ij})$ holds for every choice ϕ^k_{ij} in $(G, \mathbb{F})$.*

FSFs are reduced to the one attribute `sym` whose labels are the collections of symmetric choices. Since the symmetry of choices is an equivalence relation, the frame $\mathbb{F}'$ is always well-defined. Obviously, the reduction cannot destroy the symmetries of choices—choices that have been symmetric before a reduction remain symmetric after the reduction of the frame. Thus, a symmetry invariant strategy combination of the reduced FSF is always a symmetry invariant strategy combination of the original FSF.

Lemma 4.7. *If strategy combination b is symmetry invariant in the reduction $r(G, \mathbb{F})$ of a FSF $(G, \mathbb{F})$, then it is symmetry invariant in $(G, \mathbb{F})$.*

It is possible to apply this reduction repeatedly to the resulting FSFs. Except for the initial reduction, each following reduction step does not increase the number of labels in the frame. As the number of labels is finite, the reduction process eventually becomes stationary after a finite number of steps. This stationary FSF is called the completely reduced form of the FSF and is denoted $\bar{r}\,(G, \mathbb{F})$.

5. Applications

5.1. **Rationalizing the choice of the salient.** The perception of salience in matching games $G_m[n]$ can be modelled by a frame $\mathbb{F}_{\mathtt{S}}[n]$ straightforwardly. The language part simply consists of $\Lambda_{\mathtt{S}} = \{\mathtt{salient}, \mathtt{non\text{-}salient}\}$ and $\mathfrak{A}_{\mathtt{S}} = \{\mathtt{salience}\}$. Without loss of generality, the label function of both players i can be set to $\ell^{\mathtt{S}}_{i\mathtt{salience}}(\phi^1_i) = \mathtt{salient}$ and $\ell^{\mathtt{S}}_{i\mathtt{salience}}(\phi^j_i) = \mathtt{non\text{-}salient}$ for $j \neq 1$.

Obviously, for $n \geq 3$, on the one hand, the strategies ϕ^j_i for $j \neq 1$ are symmetric to each other in the FSF $(G[n], \mathbb{F}_{\mathtt{S}}[n])$. On the other hand, the strategies ϕ^1_1 and ϕ^1_2 are also symmetric. However, the strategies in each of the groups mentioned are not symmetric to the strategies in the other group. This is so because a symmetry of $(G[n], \mathbb{F}_{\mathtt{S}}[n])$ that would constitute the symmetry of ϕ^1_1 and ϕ^2_1 would have to map the label `salient` to the label `non-salient` and vice versa. The labels `salient` and `non-salient` are assigned to a different number of strategies but, according to lemma 4.4 this is impossible. Thus, in every symmetry invariant strategy, all choices except for ϕ^1_i have to be assigned the same probability. The only payoff dominant symmetry invariant equilibrium is the choice of the salient option by both players. So far, symmetry invariance together with payoff dominance leads to the same result as Gauthier's approach.

Things turn out to be different for $n = 2$. In this case, both the label `salient` and the label `non-salient` are assigned to just two choices—to one choice of each player. Therefore, both of a player's choices could be—and actually are—symmetric. The only symmetry invariant strategy combination is uniform randomization over both of a player's choices. Thus, common salience of one choice is not sufficient to justify the players' choosing this salient choice. This departure from Gauthier is fully in line with the criticism raised by Provis (1977) and Gilbert (1990). The non-salient choice from exactly two choices is—in some sense—to the same extent 'salient' by its being the only choice that is not salient as the salient choice by its being the salient one.

5.2. **FSF representations of VUGs.** Consider VUGs based on matching games $G_m[n]$. The player set $N_{\mathbf{R}}$ of the standard form representation contains separate players for each repertoire r of an original player i, $N_{\mathbf{R}} = \{ir : ir \in N_m \times \mathbf{R}\}$. These players are also called repertoire agents. A player's repertoire r determines his set of (subjectively) perceived pure strategies $\Phi^A_{ir} = A(r) \subset B^m_i$. $\Phi^{A\mathbf{R}}[n]$ denotes the set of pure-strategy combinations $\times_{ir \in N_{\mathbf{R}}} \Phi^A_{ir}$. The repertoire agents' expected payoff functions are defined as

$$H^V_{ir}(b) = \sum_{r' \in \mathbf{R}} V(r'|r) H^m_i(b_{ir} b_{-ir'}) = \sum_{r' \subseteq r} V(r'|r) H^m_i(b_{ir} b_{-ir'}).$$

Let H^V denote the collection of all players' payoff functions $(H^V_{ir})_{N_{\mathbf{R}}}$. Consider the player sets $C(r)$ containing all type agents ir' such that r' is a subrepertoire of r. Each set $C(r)$ forms a cell of standard form representation $(\Phi^{A\mathbf{R}}[n], H^V)$, and $C(\emptyset)$ forms an elementary cell. By definition 3.4 and 3.5, cell and truncation consistency require solving the game step by step. First, the solutions of the elementary cells have to be worked out. After fixing these players at their solution, the pairs $\{1r, 2r\}$ of one-family repertoire agents become elementary cells. This procedure is repeated until the solution of each repertoire agent is determined. The decomposition and step-by-step solution ensures that the solution of a repertoire agent is not influenced by the families outside of his repertoire.

In standard form representations of VUGs, the players' perception of the game is directly incorporated in the strategy sets Φ^A_{ir} of the repertoire agents. FSF representations of VUGs use a different method. They leave the original strategy sets unchanged and put the information contained in the system of families into the frame.

The VUG $(\Phi^{A\mathbf{R}}[n], H^V)$ is represented by the FSF $((\Phi^{\mathbf{R}}[n], H^V), \mathbb{F}_{\mathbf{F}})$, where $\Phi^{\mathbf{R}}[n] = \times_{ir \in N_{\mathbf{R}}} \Phi_{ir}$ and $\Phi_{ir} = \Phi^m_i$. The frame $\mathbb{F}_{\mathbf{F}}$ consists of the attribute set $\mathfrak{A}_{\mathbf{F}}$ that contains one attribute $\mathfrak{a}_F$ for each family F, the label set $\Lambda_{\mathbf{F}}$ comprising the concepts of all families as labels and an additional label `player_is_not_aware_of_this_family`, and the label functions $\ell^{\mathbf{F}}_{ir\mathfrak{a}_F}$. If the family F of an attribute $\mathfrak{a}_F$ is contained in the repertoire r of player ir, the label function $\ell_{ir\mathfrak{a}_F}$ simply assigns the appropriate concept label to this player's strategies. In the case that F lies outside r, all of the players strategies get the label `player_is_not_aware_of_this_familiy`.

The modified versions of cell and truncation consistency for FSFs (definition 4.3 and 4.4) require a step-by-step solution beginning with the no-family repertoire agents. Since a repertoire agent's labelling leaves his actions indistinctive as to family attributes outside of his repertoire—all strategies receive the label `player_is_not_aware_of_this_family`, these families have no impact on his solution. In this respect, FSF representations of VUGs lead to the same implications as the VUGs themselves.

The main difference between VUGs and their FSF representations lies in the way they restrict the players' strategy sets. While VUGs directly restrict the repertoire agents' strategy sets by the coverage function, it is the symmetry invariance requirement that actually restricts the players' strategic possibilities in FSFs. As it will be shown, both approaches often lead to the same solution.

5.3. **Reduction of FSFs.** Consider the following solution function for the games $((\Phi^{\mathbf{R}}[n], H^V), \mathbb{F}_{\mathbf{F}})$:

1. The solution function is cell and truncation consistent.
2. The solution is symmetry invariant.
3. The solution of indecomposable FSFs is the solution of their completely reduced form.
4. The solution of completely reduced indecomposable FSFs is the (unique) payoff dominant and risk dominant symmetry invariant equilibrium.

The risk dominance criterion used in this connection takes the following form. As it happens, the risk comparison of strategy combinations has to be done in two-player games only. For each player, the strategy set is restricted to the two strategies of the equilibria to compare. Then, HS's (pp. 86) risk dominance criterion for 2×2-games can be applied.

In this solution, symmetry invariance plays the role of Janssen's condition (*iii*), reduction the role of condition (*iv*)—versions of the *principle of insufficient reason*—and payoff dominance the role of condition (*v*)—the *principle of individual team member rationality.* In contrast to Janssen, the solution concept itself requires the choice of an equilibrium. But as the proof of theorem 5.1 shows, payoff dominance already implies the selection of an equilibrium. Cell and truncation consistency are rather technical requirements that are due to the specific properties of FSF representations. Risk dominance secures that there is always a (unique) solution. Without this requirement, however, arguments similar to those in the proof of theorem 5.2 prove generic uniqueness.

The following two theorems are analogous to theorem 1 and 2 of Janssen. The proofs are found in the appendix.

Theorem 5.1. *For every FSF* $((\Phi^{\mathbf{R}}[n], H^V), \mathbb{F}_{\mathbf{F}})$ *the solution function with the above properties exists.*

For given n and $\mathbf{F}$, each FSF $((\Phi^{\mathbf{R}}[n], H^V), \mathbb{F}_{\mathbf{F}})$ is characterized by the probabilities of the families $V(F)$. Thus, this class of FSFs is represented by a subset of $\mathbb{R}^{|\mathbf{F}|}$, the set $\mathcal{V} = \bigtimes_{F\in\mathbf{F}}(0,1]$. A proposition about the class of FSFs $((\Phi^{\mathbf{R}}[n], H^V), \mathbb{F}_{\mathbf{F}})$ holds generically, if there is a null set $\mathcal{N}$ of $\mathbb{R}^{|\mathbf{F}|}$ such that this proposition holds for all FSFs characterized by a $V \in \mathcal{V}\backslash\mathcal{N}$.

Theorem 5.2. *In generic cases, for all FSFs* $((\Phi^{\mathbf{R}}[n], H^V), \mathbb{F}_{\mathbf{F}})$ *holds: If* $F \in r$ *and* $H_{i\{F\}}(b^*_{\{F\}}) > \frac{1}{n}$, *then* $H_{ir}(b^*_r) > \frac{1}{n}$.

5.4. **Label procedures.** The games considered by Sugden can be modelled by FSF representations of Bayesian forms whose types correspond to the different label functions (label function agents), and whose type pairs are distributed like the pairs of label functions. Their frames would have to contain all of the labels assigned by the original label functions and the only attribute `label`. The FSFs' label functions would assign the same labels to the strategies of a label function agent as the original label function that he represents.

5.5. **Blockmarking and Choose an Object reconsidered.** In this section, Blockmarking and Choose an Object are reconsidered. The games are transformed into FSFs and the application of symmetry invariance is demonstrated. It is shown that symmetry invariance mostly leads to the same intuitive solutions as Bacharach's and Janssen's approaches. Moreover, it is shown that within the FSF framework some subtle kinds of salience can be explained endogenously.

In Blockmarking 1, randomization is the only symmetry in variant strategy for both players. The same arguments as in section 5.1 lead to taking the red block in Blockmarking 2. Blockmarking 3 can be modelled as FSF representation of a VUG based on G_m[20]. It is assumed that $V(\texttt{color}) = 1$ and $V(\texttt{grain}) = v$. Therefore, only two repertoires have to be considered—$\{\texttt{color}\}$ and $\{\texttt{color}, \texttt{grain}\}$. The labelling of the repertoire agents is given by

r	$\{\texttt{color}, \texttt{grain}\}$		
$\ell_{i\cdot}(\phi_{ir}^k)$	ϕ_{ir}^1, ϕ_{ir}^2	ϕ_i^3	$\phi_{ir}^4 \cdots \phi_{ir}^{20}$
color	red	yellow	yellow
grain	straight	wavy	straight

r	$\{\texttt{color}\}$		
$\ell_{i\cdot}(\phi_{ir}^k)$	ϕ_{ir}^1	ϕ_i^3	$\phi_{ir}^4 \cdots \phi_{ir}^{20}$
color	red	yellow	yellow
grain	not_realized	not_realized	not_realized

Both $\{\texttt{color}\}$-agents form an elementary cell of which the solution is randomizing over the red blocks. After fixing the $\{\texttt{color}\}$-agents at this solution, the payoff matrix for repertoire agent $i\{\texttt{color}, \texttt{grain}\}$ is

	ϕ_{-ir}^1	ϕ_{-ir}^2	ϕ_{-ir}^3	ϕ_{-ir}^4	$\cdots$	ϕ_{-ir}^{20}
ϕ_{ir}^1	$\frac{1}{2}(1-v)+v$	$\frac{1}{2}(1-v)$	$\frac{1}{2}(1-v)$	$\frac{1}{2}(1-v)$	$\cdots$	$\frac{1}{2}(1-v)$
ϕ_{ir}^2	$\frac{1}{2}(1-v)$	$\frac{1}{2}(1-v)+v$	$\frac{1}{2}(1-v)$	$\frac{1}{2}(1-v)$	$\cdots$	$\frac{1}{2}(1-v)$
ϕ_{ir}^3	0	0	v	0	$\cdots$	0
ϕ_{ir}^4	0	0	0	v	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	0
ϕ_{ir}^{20}	0	0	0	0	0	v

.

By lemma 4.4 the labels are not symmetric. Thus, the strategies within the groups $\{\phi_{ir}^1, \phi_{ir}^2\}$, $\{\phi_{1r}^3, \phi_{2r}^3\}$, and $\{\phi_{ir}^4 \cdots \phi_{ir}^{20}\}$ only are symmetric. As in the VUG, for $v > \frac{1}{2}$ taking the wavy yellow block, and for $v < \frac{1}{2}$ picking a red block is payoff dominant.

Choose an Object 1 can be represented by the FSF $(G_m[6], \mathbb{F})$ where $\mathfrak{A} = \{$`color`,`shape`$\}$, $\Lambda = \{$`white`, `red`, `yellow`, `round`, `square`$\}$ and

$\ell_{i\cdot}(\cdot)$	ϕ_i^1	ϕ_i^2	ϕ_i^3	ϕ_i^4	ϕ_i^5	ϕ_i^6
color	white	white	red	yellow	yellow	yellow
shape	round	square	round	square	round	round

.

By lemma 4.5, both attributes are not symmetric to each other. Thus, by lemma 4.4, none of the labels is symmetric to any of the other labels. Therefore, ϕ_i^5 and ϕ_i^6 are the only symmetric choices. Symmetry invariance in FSFs leaves the players with four equally good payoff dominant equilibria—both players take the same one of the first four objects.

This ambiguity can be overcome by the complete reduction of the FSF (see section 4.5). After the first reduction step, the label set $\{\{\phi_1^1, \phi_1^2, \phi_1^3, \phi_1^4, \phi_2^1, \phi_2^2, \phi_2^3, \phi_2^4\}, \{\phi_1^5, \phi_1^6, \phi_2^5, \phi_2^6\}\}$ is obtained. By lemma 4.4, both labels cannot be symmetric, and the reduction process becomes stationary. Thus, symmetry invariance requires both players to put the same probability on the first four strategies and to put the same probability on the last two strategies. Finally, payoff dominance selects randomizing over the two yellow balls as the solution.

Considering this game as a VUG, its FSFs representation leads to the same solution as Janssen's approach. Obviously, the solution for the repertoire agents with repertoire $\{$`color`$\}$ is choosing the red ball, and for repertoire $\{$`shape`$\}$ randomizing over both cubes. If both families are equally probable but not certain, taking the red ball is payoff dominant.

Choose an Object 2 can be represented by the FSF $(G_m[6], \mathbb{F})$ where $\mathfrak{A} = \{$`color`,`shape`$\}$, $\Lambda = \{$`white`, `green`, `red`, `yellow`, `round`, `square`$\}$ and

$\ell_{i\cdot}(\cdot)$	ϕ_i^1	ϕ_i^2	ϕ_i^3	ϕ_i^4	ϕ_i^5	ϕ_i^6
color	white	green	red	yellow	yellow	yellow
shape	round	round	round	square	round	round

.

By lemma 4.5, both attributes are not symmetric to each other, and by lemma 4.4, only the labels `white`, `green`, and `red` can be symmetric to each other. Thus, the choices ϕ_i^1, ϕ_i^2 and ϕ_i^3, and the choices ϕ_i^5 and ϕ_i^6 are symmetric. Clearly, the payoff dominant symmetry invariant equilibrium is to take the yellow cube.

Players that are aware of the objects' color and shape will frame Choose an Object 3 by $\mathbb{F}$ such that $\Lambda = \{$`white`, `red`, `round`, `square`$\}$, $\mathfrak{A} = \{$`color`, shape} and

$\ell_{i\cdot}(\cdot)$	ϕ_i^1	ϕ_i^2	ϕ_i^3
color	white	white	red
shape	round	square	round

.

Two symmetries (f, ξ) and (f', ξ') of $(G[3], \mathbb{F})$ are defined by the mappings

ϕ_i^k	ϕ_1^1	ϕ_1^2	ϕ_1^3	ϕ_2^1	ϕ_2^2	ϕ_2^3
$f_i(\phi_i^k)$	ϕ_1^1	ϕ_1^3	ϕ_1^2	ϕ_2^1	ϕ_2^3	ϕ_2^2

ϕ_i^k	ϕ_1^1	ϕ_1^2	ϕ_1^3	ϕ_2^1	ϕ_2^2	ϕ_2^3
$f'_i(\phi_i^k)$	ϕ_2^1	ϕ_2^2	ϕ_2^3	ϕ_1^1	ϕ_1^2	ϕ_1^3

,

$\mathfrak{a}$	color	shape
$\xi'_{\mathfrak{A}}(\mathfrak{a})$	color	shape

λ	white	red	round	square
$\xi'_{\Lambda}(\lambda)$	white	red	round	square

,

$\mathfrak{a}$	color	shape
$\xi_{\mathfrak{A}}(\mathfrak{a})$	shape	color

λ	white	red	round	square
$\xi_{\Lambda}(\lambda)$	round	square	white	red

.

Therefore, both players' last two choices (ϕ_i^2, ϕ_i^3) are symmetric to each other, and their first choices (ϕ_1^1, ϕ_2^1) are also symmetric. However, there is no symmetry that maps one of the last two choices to the first one. If such a symmetry would exist, one of the labels `red` or `square` would have to be mapped to one of the labels `white` or `round`. Since the first two labels are assigned to less choice-attribute pairs than the latter, this is in contradiction with lemma 4.4. Thus, symmetry invariance requires the players to take (a) the same strategy where (b) the white cube and the red ball get the same probability. Choosing the white ball—the salient option—is the unique payoff dominant symmetry invariant equilibrium of this FSF.

This example shows that the introduced notion of symmetry invariance does not simply account for differences between the denotations of the choices, but is able to detect subtle similarities in the structure of the labels. While Janssen has to introduce, in fact, the salience of the white ball exogenously, the FSF approach is able to derive its salience from the given description of the game. In doing so, symmetry invariance endogenizes salience to some extent.

5.6. **Meaning of labels.** It has been emphasized that symmetry invariance does not capture intrinsic meanings that the players attach to the labels. This is explained with some examples. Consider Choose an Object with five different colored but otherwise identical cubes. Suppose one of the cubes—the red one—is perceived as strikingly colored. Nevertheless, in a frame with the attribute `color` only, FSF symmetry prescribes randomization. The meaning of red as a striking color does not influence the solution. Only if the property of being striking is explicitly modelled with a separate attribute such as `striking`, can choosing the red cube be rationalized within the FSF framework.

	ϕ_2^1	ϕ_2^2
ϕ_1^1	1, −1	−1, 1
ϕ_1^2	−1, 1	1, −1

A

	`left`	`right`
`top`	1, −1	−1, 1
`down`	−1, 1	1, −1

B

	`heads`	`tails`
`heads`	1, −1	−1, 1
`tails`	−1, 1	1, −1

C

Figure 1. Matching Pennies

Figure 1 shows a standard form of Matching Pennies (A) and two related FSFs (B,C). Besides the identity mapping, there are three other standard form symmetries:

1. $f_1(\phi_1^1) = \phi_1^2, f_1(\phi_1^2) = \phi_1^1, f_2(\phi_2^1) = \phi_2^2, f_2(\phi_2^2) = \phi_2^1$,
2. $f_1'(\phi_1^1) = \phi_2^2, f_1'(\phi_1^2) = \phi_2^1, f_2'(\phi_2^1) = \phi_1^1, f_2'(\phi_2^2) = \phi_1^2$,
3. $f_1''(\phi_1^1) = \phi_2^1, f_1''(\phi_1^2) = \phi_2^2, f_2''(\phi_2^1) = \phi_1^2, f_2''(\phi_2^2) = \phi_1^1$.

Thus, both players are symmetric—the players cannot distinguish their position without referring to the game theorist's strategy labels. Oh (1995) argues that in Matching Pennies, if players attach meaning to the labels as in (C), both players are no longer symmetric—in (C), player 1 can be referred to as the player who wins if the same label is chosen by both players, and in (B), player 1 could be referred to as the ⟨`top`, `left`⟩-⟨`down`, `right`⟩-winner. In some intuitive sense, the players are not symmetric either in (B) or (C).

Considering the strategy labels in (B) and (C) as labels of the only attribute `label`, the FSF symmetry leads to different results. While in (C) the players remain non-symmetric, in (B) the players are symmetric like in (A). Since FSF isomorphisms involve a bijective mapping of the label sets, this is not surprising. Both symmetries that constitute the symmetry of the players—the second and the third—rest on mappings of the choices that switch the choices of one player and leave the order of the other player's choices unchanged. On the one hand, in (B), both players' choices are labelled independently. Hence the renaming of the choices is compatible with the choices' labelling. On the other hand, in (C), the players have a common way of labelling the choices. No matter how the language is translated, player 1 wins if both players take choices with the same label, and player two wins if different labels are chosen. It is the whole structure of the choices' labels rather than the individual label that matters to the players. The common labelling actually destroys the symmetry of the players in (C).

5.7. **Labelling players.** Though the label functions of a frame explicitly assign multi-dimensional labels to the agents' choices only, it is possible to denote the agents and players themselves. This can be done by introducing attributes that characterize agents or players, like `color_of_agent` or `sex_`

ϕ_3^1	ϕ_2^1	ϕ_2^2
ϕ_1^1	0, 0, 0	1, 1, 1
ϕ_1^2	1, 1, 1	0, 0, 0

ϕ_3^2	ϕ_2^1	ϕ_2^2
ϕ_1^1	1, 1, 1	0, 0, 0
ϕ_1^2	0, 0, 0	0, 0, 0

FIGURE 2. Standard form for Take Two Black Balls

`of_player`, and adding appropriate labels and label functions. Since agent or player attributes should not distinguish the choices, these label functions are to assign the same label to all choices of an agent or player.

Reconsider Take Two Black Balls. Figure 2 shows a standard form for this game. Player 1 is row chooser, player 2 column chooser, and player 3 matrix chooser. The payoffs start with player 1 bottom left and end with player 3 top right. An appropriate frame for players that realize the sex of players is $\mathbb{F}$ such that $\Lambda = \{\texttt{female}, \texttt{male}\}$, $\mathfrak{A} = \{\texttt{sex\_of\_player}\}$ and

$\ell_{i\cdot}(\cdot)$	ϕ_1^1	ϕ_1^2	ϕ_2^1	ϕ_2^2	ϕ_3^1	ϕ_3^2
`sex_of_player`	`female`	`female`	`female`	`female`	`male`	`male`

.

According to lemma 4.4 the labels cannot be symmetric. Therefore, the male player is not symmetric to any of the female players, but the female players are symmetric to each other. Symmetry invariance implies that both women take the same strategy. Payoff dominance, then, leads to the intuitive solution—the women choose the black ball, and the man chooses the white ball.

5.8. **History in games.** Oh (1995, pp 7-34) extends extensive forms by adding a history structure. For each information set, the history is the set of the preceding moves, a player's history is the collection of his information sets' histories, and the history of the game is the collection of all players' histories. In order to exploit this additional structure, Oh modifies the notion of a symmetry of extensive forms by adding a one-to-one mapping of the game's history set onto itself and adapts the notions of cell and truncation consistency. Although not (explicitly) modelled, sensible use of these symmetries requires the introduction of the players' labelling of the moves. In addition, it requires that the history sets contain the players' labels of the previous moves and not (only) the game theorist's. Otherwise the history would not restrict the set of symmetries as was intended.

These shortcomings can be overcome within the FSF framework. Consider the standard form representation $G = (\Phi, H)$ of an extensive form. Assume, for the sake of parsimony, that the players originally distinguish their choices with respect to one attribute only. This can be expressed with a FSF $(G, \mathbb{F})$ such that the attribute set $\mathfrak{A}$ contains the only attribute `label`, and the label set Λ contains just the labels assigned by the label functions $\ell_{ij\texttt{label}}$. In order to formalize the history, $\mathfrak{A}$ is extended by attributes $\texttt{pre?\_}\phi_{ij}^{k}$ for every choice ϕ_{ij}^{k} in G, Λ is extended by the label no, and the label functions $\ell_{ij\texttt{pre?\_}\phi_{ij}^{k}}$ are added. These additional label functions are defined such that

$$\ell_{i'j'\texttt{pre?\_}\phi_{ij}^{k}}(\phi_{i'j'}^{k'}) = \begin{cases} \ell_{ij\texttt{label}}(\phi_{ij}^{k}) & : \ \phi_{ij}^{k} \text{ precedes information set } i'j' \\ \texttt{no} & : \ \phi_{ij}^{k} \text{ does not precede } i'j' \end{cases}.$$

The following example demonstrates how this kind of history representation, together with the requirements of cell and truncation consistency, symmetry invariance, and payoff dominance, determines solutions of repeated coordination games.

Consider the twice repeated matching game $G_m[3]$. It is assumed that the players move simultaneously at each stage and learn the other players first action and their payoff after the first stage. Hence, besides the whole game, there are 9 proper subgames that form elementary cells of the FSF representation. According to cell and truncation consistency, first the solutions of these cells are to be determined. Two cases are possible: either (a) the players got coordinated in the first stage or (b) they failed to coordinate.

For example, the history of the elementary cell (information set) 2 succeeding coordination at the label combination $\langle \texttt{A}, \texttt{a} \rangle$ can be represented by the frame below:

$\ell_{\cdot 2 \cdot}(\phi_{\cdot 2}^{\cdot})$		ϕ_{12}^{1}	ϕ_{12}^{2}	ϕ_{12}^{3}	ϕ_{22}^{1}	ϕ_{22}^{2}	ϕ_{22}^{3}
label		A	B	C	a	b	c
pre?_	ϕ_{11}^{1}	A	A	A	A	A	A
	ϕ_{21}^{1}	a	a	a	a	a	a
	other	no	no	no	no	no	no

Obviously, the actions labelled A and a are symmetric to each other, but they are not symmetric to the actions labelled B, C, b, and c that, in turn, are symmetric to each other. In the same way, a cell succeeding discoordination, for example, at the label combination $\langle \texttt{B}, \texttt{c} \rangle$ is modelled.

$\ell_{\cdot 6 \cdot}(\phi_{\cdot 6}^{\cdot})$		ϕ_{16}^{1}	ϕ_{16}^{2}	ϕ_{16}^{3}	ϕ_{26}^{1}	ϕ_{26}^{2}	ϕ_{26}^{3}
label		A	B	C	a	b	c
pre?_	ϕ_{11}^{1}	B	B	B	B	B	B
	ϕ_{21}^{1}	c	c	c	c	c	c
	other	no	no	no	no	no	no

Again, both players' second and third strategies are symmetric, whereas their first strategies are not. Hence, in both cases, symmetry invariance combined with payoff dominance ensures coordination in the second stage—in the first case, a player repeats his coordinating action from the first stage, and in the second case, he takes the action that was not taken in the first stage and that would not have produced coordination in the first stage. This result is in line with Crawford & Haller (1990).

6. Conclusion

FSFs generalize the formalization of the players' description of a game—of both the strategies and the players themselves—in a similar way as it is proposed by Bacharach, Janssen and Sugden. This framework is applicable to general n-player games and to extensive and Bayesian forms via their standard form representation, and implicitly allows the description of the players. The related notion of symmetry invariance rests upon the simple, intuitively plausible assumption of language invariance and suggests an alternative possibility of how players could make use of the additional information provided by a frame. On the one hand, symmetry invariance combines the several features that characterize the strategies and players in a game. On the other hand, it does not simply recognize differences in these descriptions but also accounts for more subtle similarities. It is shown that this approach is amenable to extensions such as the reduction of the frame and some notion of risk dominance.

7. Appendix

Proof. (Theorem 5.1) In view of the theorems 4.2 and 4.6, and lemma 4.7 the first three properties are compatible. It remains to show that payoff and risk dominance determine a unique solution.

According to cell and truncation consistency, the solution function is to be applied to the cells formed successively by the player sets $\{1r, 2r\}$. First it is shown that for a cell $\{1r, 2r\}$, the strategies $\phi^k_{1r} \in \Phi_{1r}$ and $\phi^k_{2r} \in \Phi_{2r}$ are symmetric. For the empty repertoire r, $|r| = 0$, the related FSF cell is simply the original matching game and all strategies are assigned the label `player_is_not_aware_of_this_attribute` with respect to all attributes $\mathfrak{a}_F$. Clearly, all strategies are symmetric and the reduction becomes stationary after the first step. Thus, both players' solution for the empty repertoire is uniform randomization over the whole strategy set.

Suppose all repertoires r', $|r'| < |r|$, form cells $\{1r', 2r'\}$ such that the players are symmetric. Note, this also implies that a solution requires both players to take the same strategy. The players' payoffs in the cell formed by $\{1r, 2r\}$ are given by

$$\begin{aligned} H^*_{ir}(b_{ir}b_{-ir}) &= \sum_{r' \subsetneq r} V(r'|r) H^m_i(b_{ir}b^*_{-ir'}) + V(r|r) H^m_i(b_{ir}b_{-ir}) \\ H^*_{-ir}(b_{ir}b_{-ir}) &= \sum_{r' \subsetneq r} V(r'|r) H^m_{-i}(b^*_{ir'}b_{-ir}) + V(r|r) H^m_{-i}(b_{ir}b_{-ir}) \end{aligned}$$

where $b^*_{ir'}$ and $b^*_{-ir'}$ denote the solutions of the repertoires r'. As matching games $H^m_{-i} = H^m_i$ and $H^m_i(b_i, b_{-i}) = H^m_i(b_{-i}, b_i)$, and as it is assumed that $b^*_{ir'} = b^*_{-ir'}$, $H^*_{ir}(b_{ir}, b_{-ir}) = H^*_{-ir}(b_{-ir}, b_{ir})$ holds. Since, in fact, both players $1r$ and $2r$ have the same label functions, the players are symmetric, and also are the related strategies ϕ^k_{1r} and ϕ^k_{2r}. By induction, this holds for all repertoires.

Symmetry invariance then leads both players to choose the same strategy. This implies that the players receive the same payoff and all symmetry invariant equilibria are Pareto rankable, but there may be several symmetry invariant equilibria that yield the same maximum payoff. For these cases, it is now shown that risk dominance always selects a unique strategy combination.

For $|r| = 0$ the cell solution is trivially unique by symmetry alone. Suppose the solutions for all repertoires $r^{'}$, $|r^{'}| < |r|$, are unique. Thus, the payoff functions for the cell $\{1r, 2r\}$ are uniquely determined. After complete reduction, the labels of the resulting frame partition both players' strategy sets in the same way. Let $\Omega_r = \{\omega^k_r\}$ denote this partition of the players strategy sets. Let $b_{ir}(\omega^k_r)$ be the strategy of player ir that assigns the same probability to all strategies in ω^k_r and the probability 0 to all strategies outside ω^k_r. Let $b_r(\omega^k_r) = b_{ir}(\omega^k_r)b_{-ir}(\omega^k_r)$. All symmetry invariant strategies $\hat{b}_{ir}$ are mixtures of the $b_{ir}(\omega^k_r)$ and can be characterized by a collection of non-negative real numbers α^k_i summing up to 1 where α^k_i stands for player ir's probability of $b_{ir}(\omega^k_r)$. In a symmetry invariant strategy combination $\hat{b}_r = \hat{b}_{ir}\hat{b}_{-ir}$ holds $\alpha^k_i = \alpha^k_{-i} = \alpha^k$ and therefore (see Janssen 1995, p. 21)

$$H^*_{ir}(\hat{b}_r) = \sum_{\omega^k_r \in \Omega_r} \left[\alpha^k H^*_{ir}(b(\omega^k_r)) - V(r|r)\alpha^k(1-\alpha^k)H^m_i(b_r(\omega^k_r))\right].$$

Since the second term in the sum is always positive, shifting the probability mass completely to one of the strategy combinations $b\left(\omega^k_r\right)$ that yield the highest payoff $H^*_{ir}(b\left(\omega^k_r\right))$ raises the payoff of $\hat{b}$. Therefore a Pareto best symmetry invariant strategy combination is of the form $b_r\left(\omega^k_r\right)$.

Next, it is shown that a Pareto best strategy combination $b_r(\omega^*_r)$ is an equilibrium. Let $\phi^l_{ir}, \phi^m_{ir} \in \omega^n_r$. Then, ϕ^l_{ir} and ϕ^m_{ir} are symmetric, and there is a symmetry f that maps ϕ^l_{ir} to ϕ^m_{ir}. Let $\hat{b}_{-ir}$ be part of a symmetry invariant strategy combination. By definition of symmetries and symmetry invariance we have

$$\begin{aligned} H^*_{ir}(\phi^l_{ir}\hat{b}_{-ir}) &= H^*_{ir}(f(\phi^l_{ir}\hat{b}_{-ir})) = H^*_{ir}(f_{ir}\phi^l_{ir})f_{-ir}(\hat{b}_{-ir})) \\ &= H^*_{ir}(\phi^m_{ir}\hat{b}_{-ir}). \end{aligned} \tag{2}$$

It holds

$$H^*_{ir}(\phi^l_{ir}b_{-i}(\omega^k_r)) = \sum_{r^{'}\subsetneq r} V(r^{'}|r)H^m_i(\phi^l_{ir}b^*_{-ir^{'}}) + V(r|r)H^m_i(\phi^l_{ir}b_{-ir}(\omega^k_r)).$$

Since either $\phi^l_{ir}, \phi^m_{ir} \notin \omega^k_r$ or $\phi^l_{ir}, \phi^m_{ir} \in \omega^k_r$ holds, by (2) follows

$$\sum_{r^{'}\subsetneq r} V(r^{'}|r)H^m_i(\phi^l_{ir}b^*_{-ir^{'}}) = \sum_{r^{'}\subsetneq r} V(r^{'}|r)H^m_i(\phi^m_{ir}b^*_{-ir^{'}}). \tag{3}$$

Since $H_i^m(\phi_{ir}^m b_{-ir}\left(\omega_r^k\right)) = H_i^m(b_{ir}\left(\omega_r^n\right) b_{-ir}\left(\omega_r^k\right))$, we have

$$\begin{aligned} H_{ir}^*(b_{ir}\left(\omega_r^n\right) b_{-ir}\left(\omega_r^k\right)) &= \sum_{\phi_{ir}^l \in \omega_r^n} |\omega_r^n|^{-1} \sum_{r' \subsetneq r} V(r'|r) H_i^m(\phi_{ir}^l b_{-ir'}^*) \\ &\qquad + V(r|r) H_i^m(\phi_{ir}^m b_{-ir}(\omega_r^k)) \end{aligned}$$

and by (3)

$$\begin{aligned} H_{ir}^*(b_{ir}\left(\omega_r^n\right) b_{-ir}\left(\omega_r^k\right)) &= \sum_{r' \subsetneq r} V(r'|r) H_i^m(\phi_{ir}^l b_{-ir'}^*) \\ &\qquad + V(r|r) H_i^m(\phi_{ir}^m b_{-ir}(\omega_r^k)) \\ &= H_{ir}^*(\phi_{ir}^m b_{-ir}(\omega_r^k)). \qquad (4) \end{aligned}$$

Thus, it is sufficient to show that $b_r\left(\omega_r^*\right)$ is an equilibrium with respect to the strategies that are symmetry invariant after the reduction. Suppose that $\omega_r^* \neq \omega_r^k$ and $H_{ir}^*(b_{ir}\left(\omega_r^*\right) b_{-ir}\left(\omega_r^*\right)) < H_{ir}^*(b_{ir}(\omega_r^k) b_{-ir}\left(\omega_r^*\right))$. Since $H_i^m(b_{ir}(\omega_r^k) b_{-ir}\left(\omega_r^*\right)) = 0$ and $V(r|r) H_i^m(b_r(\omega_r^k)) > 0$, it follows that

$$H_{ir}^*(b_r(\omega_r^*)) < \sum_{r' \subsetneq r} V(r'|r) H_i^m(b_{ir}(\omega_r^k) b_{-ir'}^*) < H_{ir}^*(b_r(\omega_r^k)),$$

a contradiction to $b_r\left(\omega_r^*\right)$ being Pareto best. Therefore $b_r\left(\omega_r^*\right)$ is an equilibrium.

Finally, it is shown that there cannot exist two different Pareto best equilibria of the form $b_r(\omega_r^k)$ such that one does not risk dominate the other. Consider two different Pareto best equilibria $b_r(\omega_r^k)$ and $b_r(\omega_r^n)$. $b_r(\omega_r^k)$ is risk dominant iff

$$H_{ir}^*(b_{ir}(\omega_r^k) b_{-ir}\left(\omega_r^n\right)) > H_{ir}^*(b_{ir}\left(\omega_r^n\right) b_{-ir}(\omega_r^k)),$$

or, equivalently, iff $|\omega_r^k| > |\omega_r^n|$. Thus, the risk dominance of Pareto best equilibria is transitive. Suppose $b_r(\omega_r^k)$ and $b_r(\omega_r^n)$ do not risk dominate each other,

$$H_{ir}^*(b_{ir}\left(\omega_r^n\right) b_{-ir}(\omega_r^k)) = H_{ir}^*(b_{ir}(\omega_r^k) b_{-ir}\left(\omega_r^n\right)).$$

Thus, it follows that $|\omega_r^k| = |\omega_r^n|$ and

$$\sum_{r' \subsetneq r} V(r'|r) H_i^m(b_{ir}(\omega_r^k) b_{-ir'}^*) = \sum_{r' \subsetneq r} V(r'|r) H_i^m(b_{ir}\left(\omega_r^n\right) b_{-ir'}^*).$$

By (4), for $\phi_{ir}^k \in \omega_r^k$ and $\phi_{ir}^n \in \omega_r^n$, we have

$$\sum_{r' \subsetneq r} V(r'|r) H_i^m(\phi_{ir}^k b_{-ir'}^*) = \sum_{r' \subsetneq r} V(r'|r) H_i^m(\phi_{ir}^n b_{-ir'}^*),$$

and therefore, $H_{ir}^*(\phi_{ir}^k \phi_{-ir}^k) = H_{ir}^*(\phi_{ir}^n \phi_{-ir}^n)$ and $H_{ir}^*(\phi_{ir}^k \phi_{-ir}^n) = H_{ir}^*(\phi_{ir}^n \phi_{-ir}^k)$. As by $|\omega_r^k| = |\omega_r^n|$ the labels of ϕ_{ir}^k and ϕ_{ir}^n are assigned to the same number of strategies, the mapping f that maps ω_r^k onto ω_r^n, and that maps the other strategies to themselves is a symmetry of the completely reduced cell formed by $\{1r, 2r\}$. Thus, $\phi_{ir}^k$and ϕ_{ir}^n are symmetric. Clearly, this contradicts complete reduction. Payoff and risk dominance always select a unique solution. This completes the proof. ∎

Proof. (Theorem 5.2) Note, the solution for repertoire $\{F\}$ is not influenced by the choice of V, and the carriers of $b_{\{F\}}^*$ do not comprise the whole strategy set. Suppose $F \in r$, $H_{i\{F\}}(b_{\{F\}}^*) > \frac{1}{n}$ and $H_{ir}(b_r^*) = \frac{1}{n}$. Consider the symmetry invariant strategy combination $\hat{b}_r$ such that $a_i^k = \frac{|\omega_r^k|}{n}$. Clearly, this indicates randomization over the strategy sets (see Janssen 1995, p. 23). For $H_{ir}^*(\hat{b}_r)$ we have

$$\sum_{\omega_r^k \in \Omega_r} \left[\frac{|\omega_r^k|}{n} H_{ir}^*(b(\omega_r^k)) - V(r|r) \frac{|\omega_r^k|}{n} (1 - \frac{|\omega_r^k|}{n}) H_i^m(b_r(\omega_r^k)) \right] = \frac{1}{n}.$$

Case 1 $|\omega_r^k| < n$: Hence,

$$\sum_{\omega_r^k \in \Omega_r} \left[\frac{|\omega_r^k|}{n} H_{ir}^*(b(\omega_r^k)) \right] > \frac{1}{n}.$$

Since the terms $\frac{|\omega_r^k|}{n}$ sum up to 1, there is a ω_r^k such that $H_{ir}^*(b(\omega_r^k) > \frac{1}{n}$.

Case 2 $|\omega_r^k| = n$: Let ϕ_{ir}^k be an element and ϕ_{ir}^m be not an element of the carrier of $b_{\{F\}}^*$. Since $\phi_{ir}^k, \phi_{ir}^m \in \omega_r^k, \phi_{ir}^k$ and ϕ_{ir}^m are symmetric. Thus, $H_{ir}^*(\phi_{ir}^k \phi_{-ir}^k) = H_{ir}^*(\phi_{ir}^m \phi_{-ir}^m)$ and therefore

$$\sum_{r' \subsetneq r} V(r'|r) \left[H_i^m(\phi_{ir}^k b_{-ir'}^*) - H_i^m(\phi_{ir}^m b_{-ir'}^*) \right] = 0. \tag{5}$$

Since ϕ_{ir}^k is an element and ϕ_{ir}^m is not an element of the carrier of $b_{\{F\}}^*$, $H_i^m(\phi_{ir}^k b_{-i\{F\}}^*) - H_i^m(\phi_{ir}^m b_{-i\{F\}}^*) > 0$. Hence, (5) determines a hyperplane of the $V(r'|r)$ space and therefore a null set of $\mathcal{V}$. ∎

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