CHAPTER III

Super weak isomorphism of extensive games

This Chapter has been published as "André Casajus (2006): Super weak isomorphism of extensive games, in: *Mathematical Social Sciences* 51, 107–116". Unfortunately, the published version contains a lot of misprints which appeared *after* proof-reading. The margin notes indicate the first lines of the respective pages in the published version.

Abstract [107]

It is well-known that the normal form suffices to determine some but not to determine all sequential equilibria of a game in general. How much more structure does so? In this addendum to Casajus (2003), we suggest the concept of super weak isomorphism (SWI) as an attempt to answer this question. In contrast to weak isomorphism, SWI is not sensitive to the structure of the chance mechanism and the assignment of payoffs to the individual terminal nodes. Yet, sequential equilibrium remains invariant under SWI, i.e. the structural features preserved by SWI already determine sequential equilibrium. In addition, SWI is generically equivalent to isomorphism of the agent normal form for a larger set of games than weak isomorphism.

Key Words: Symmetry, Representation, Equivalence, Sequential equilibrium, Agent normal form.

JEL classification: C72.

1. Introduction

There are games with the same agent normal (ANF) form but different sets of sequential equilibria (e.g. Kreps & Wilson 1982, Figures 2 and 13). Hence in general, the ANF does not suffice the determine *all* sequential equilibria of an extensive game. Generically, however, it does so: Generically, sequential equilibrium coincides with perfect equilibrium which can be defined via the ANF (Selten 1975). Kohlberg & Mertens (1986) show that the normal form suffices to find *some* of the sequential equilibria of an extensive game: Proper [108]
equilibria (Myerson 1978) of strategic games can be extended into sequential equilibria of extensive games with that normal form.

Which part of the structure of extensive games suffices to determine sequential equilibrium? We employ isomorphism to characterize structural features: Isomorphic games share the features implicit in the concept of isomorphism under consideration. For extensive games, there are two such concepts, strong isomorphism (Elmes & Reny 1994, Peleg et al. 1999) and weak isomorphism (WI) (Casajus 2003, henceforth CA03[1]). In addition, the Harsanyi & Selten (1988) isomorphism of the ANF (*ANF isomorphism*) or of the (reduced) normal form can be regarded as such concepts. The question then is whether sequential equilibrium is invariant under the isomorphism under consideration. Our leading example reveals that ANF isomorphism is not such a concept.

Sequential equilibrium is invariant under strong isomorphism and WI. Yet, both concepts keep (most of) the structure of extensive games. Can we do with less? We can. In this paper, we relax WI into the concept of *super weak isomorphism* (SWI) which ignores the structure of the chance mechanism while preserving the payoffs of strategy profiles. This way, the generic equivalence of WI and ANF isomorphism extends to some subset of games with a chance mechanism (Theorem 3.6). Nevertheless, sequential equilibrium remains invariant under SWI (Theorem 3.7). To enable this, SWI must preserve the sequential structure beyond the ANF. This, however, seems to be in line with Govindan & Wilson (2004) who "accept the relevance of extensive form analysis" and weaken the reduced normal form invariance requirement of Kohlberg & Mertens (1986).

This note is organized as follows: Basic definitions and notation not found in CA03 are given in the next section. In the third one, we relax WI into SWI and explore its properties. Some remarks conclude the note. The Appendix contains some proofs.

2. Basic definitions and notation

We only give the definitions and notation not given in or deviating from CA03. In order to avoid set theoretic complications, we assume that there is a set which contains all

[1]Also Chapter II of this thesis.

labels for players, pure strategies, and nodes. This way, the collections of all games and of all forms (strategic, extensive) are sets.

We set $i_0 = 0$, $I_- = I \backslash \{0\}$, $A_- := A \backslash A_0$, $H_- := H \backslash H_0$. The *reduced terminal history* of $z \in Z$ is the set $A_-(z) := A(\psi(z)) \backslash A_0$. Further,

$$Z(\mathbf{a}) := \{z \in Z | \exists \mathbf{a}_0 \in \mathbf{A}_0 : z = z(\mathbf{a}, \mathbf{a}_0)\} = \{z \in Z | A_-(z) \subseteq \{\mathbf{a}_h | h \in H_-\}\} \tag{2.1}$$

denotes the subset of Z reachable by $\mathbf{a}$.

We denote by $\mathcal{E}^{\text{nc}} \subset \mathcal{E}$ the set of games with $P_0 = \emptyset$. An *(extensive) form* γ is a tuple $(T, \lhd, I, P, H, A)$ where the constituents are defined as in $\mathcal{E}$. $\mathcal{EF}$ $(\mathcal{EF}^*, \mathcal{EF}^{\text{nc}})$ denotes the set of forms corresponding to $\mathcal{E}$ $(\mathcal{E}^*, \mathcal{E}^{\text{nc}})$. Any $\Gamma \in \mathcal{E}$ based on a fixed $\gamma \in \mathcal{EF}$ can be described by an *assignment* $\delta = (u, p) \in \mathbb{D}(\gamma) := \mathbb{U}(\gamma) \times \mathbb{W}(\gamma)$, where $\mathbb{U}(\gamma) := \mathbb{R}^{|Z||I_-|}$, $\mathbb{W}(\gamma) := \prod_{h \in H_0} \Delta_{|A_h|-1}$, $u = (u_i)_{i \in I_-}$, $u_i \in \mathbb{R}^{|Z|}$, $p = (p_h)_{h \in H_0}$, $p_h \in \Delta_{|A_h|-1}$, and where $\Delta_k \subseteq \mathbb{R}^{k+1}$, $k \in \mathbb{N}$ denotes the k-dimensional standard simplex. We then write $\Gamma = \gamma(\delta)$. A proposition on pairs of games from $\mathcal{E}' \subseteq \mathcal{E}$ based on $\mathcal{EF}' \subseteq \mathcal{EF}$ holds *generically* iff for all $\gamma' \in \mathcal{EF}'$ there is some open and dense subset $\mathcal{D}(\gamma') \subseteq \mathbb{D}(\gamma')$ such that for all $(\gamma, \bar{\gamma}) \in \mathcal{EF}' \times \mathcal{EF}'$ the proposition holds for all $(\gamma(\delta), \bar{\gamma}(\bar{\delta}))$, $\delta \in \mathcal{D}(\gamma)$, $\bar{\delta} \in \mathcal{D}(\bar{\gamma})$.

3. Super weak isomorphism [109]

3.1. Definitions. The following definition relaxes weak isomorphism by dropping its conditions related to the chance mechanism (**CPL**, **CPR**) and by weakening the other conditions accordingly. The latter is indicated by the prefix "**s**" which should be read as "superweak version of". Non-technically, a super weak isomorphism is an isomorphism of the ANF (**sISA**, **sPY**) that respects the assignment of information sets to players (**sPL**) and therefore also is an isomorphism of the normal form, for example. In addition, it preserves the RTH structure (**sPTH**).

Definition 3.1. *A* super weak isomorphism (SWI) *from $\gamma \in \mathcal{EF}$ to $\bar{\gamma} \in \mathcal{EF}$ is a bijection $r : A_- \to \bar{A}_-$ with the following properties: There are bijections $\nu : H_- \to \bar{H}_-$, $\pi : I_- \to \bar{I}_-$, and a surjective and nowhere empty correspondence $\Theta : Z \rightrightarrows \bar{Z}$ such that*

sISA $r(A_h) = \bar{A}_{\nu(h)}$ *for all* $h \in H_-$,

sPL $r(A_i) = \bar{A}_{\pi(i)}$ *for all* $i \in I_-$,

sPTH $r(A_-(z)) = \bar{A}_-(\bar{z})$ *for all* $z \in Z$ *and* $\bar{z} \in \Theta(z)$.

A SWI from $\Gamma \in \mathcal{E}$ to $\bar{\Gamma} \in \mathcal{E}$ is a SWI of the underlying forms which satisfies

sPY *for all $i \in I_-$, there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that*

$\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i(\mathbf{a}) + \beta_i$ *for all* $\mathbf{a} \in \mathbf{A}$

where $\mathbf{r} = (\mathbf{r}_{\bar{h}})_{\bar{h} \in \bar{H}_-} : \mathbf{A} \to \bar{\mathbf{A}}$, $\mathbf{r}_{\nu(h)}(\mathbf{a}) = r(\mathbf{a}_h)$.

SWI games, *SWI invariant* solution concepts and *SWI invariant* behavior-strategy profiles are defined in analogy to their WI counterparts. Obviously, r uniquely determines the bijections ν and π. In addition, **sISA** secures that the mapping $\mathbf{r}$ used in **sPY** is well-defined and bijective. $\mathbf{r}$ is extended to behavior-strategy profiles by CA03 (Equation (3.2)).

3.2. Condition sPTH. RTH determine a possibly non-atomic partition $[Z]$ of Z, $[Z] := \{[z] \,|z \in Z\}$, $z' \in [z]$ iff $A_-(z) = A_-(z')$ where $[z]$ is called the *terminal cell* containing z and $A_-([z])$ its RTH. Denote by $[Z](\mathbf{a}) \subseteq [Z]$ the set of terminal nodes reachable by $\mathbf{a}$, and by $\mathbf{A}([z]) \subseteq \mathbf{A}$ its converse, $\mathbf{a} \in \mathbf{A}([z])$ iff $[z] \in [Z](\mathbf{a})$.

The correspondence Θ from **sPTH** is unique in the following sense: By **sPTH**, we have $\bar{A}_-(\bar{z}) = \bar{A}_-(\bar{z}')$ for $\bar{z}, \bar{z}' \in \Theta(z)$ and $\Theta(z) \cap \Theta(z') = \emptyset$ if $z' \notin [z]$. Since Θ is surjective, r uniquely defines a bijection $\theta : [Z] \to [\bar{Z}]$,

$$r(A_-([z])) = \bar{A}_-(\theta([z])), \quad [z] \in [Z]. \tag{3.1}$$

In fact, **sPTH** and the existence of such a bijection θ are equivalent, and we sometimes refer to (3.1) by **sPTH**. Similar to WI, there is a characterization of **sPTH** for $\mathcal{E}^*$ involving θ. Its proof is referred to the Appendix.

Lemma 3.2. *(i)* ***sISA*** *and* ***sPTH*** *imply* ***sPTH⁻***: $[\bar{Z}](\mathbf{r}(\mathbf{a})) = \theta([Z](\mathbf{a}))$ *for all* $\mathbf{a} \in \mathbf{A}$. *(ii) In* $\mathcal{EF}^*$, ***sISA*** *and* ***sPTH⁻*** *imply* ***sPTH***.

3.3. SWI vs. weak isomorphism. The following theorem establishes the relation be- [110]
tween SWI and WI. Part (i) says that SWI weakens WI, and part (ii) says that, compared with WI, SWI just disregards the structure of the chance mechanism. While part (i) is immediate from CA03 (Lemma A.3), part (ii) follows from $|[Z](\mathbf{a})| = 1$ and $[z] = \{z\}$ for $\Gamma \in \mathcal{E}^{\mathrm{nc}}$.

Theorem 3.3. *(i) For any WI* $r : \Gamma \to \bar{\Gamma}$, *the restriction to* A_- *is a SWI* $r|_{A_-} : \Gamma \to \bar{\Gamma}$. *(ii) For* $\Gamma, \bar{\Gamma} \in \mathcal{E}^{nc}$, *any SWI* $r : \Gamma \to \bar{\Gamma}$ *also is a WI.*

The following example shows that SWI non-trivially weakens WI.[2] Casajus (2005) presents general constructions that yield SWI games: the spurious addition of chance nodes and shifting the chance mechanism to the root. Also, alternative but equivalent decompositions of a chance node's lottery do not affect SWI.

Example 3.4. Consider $\gamma, \bar{\gamma} \in \mathcal{EF}$ in Figure 3.1 where all information sets are controlled by different players. In both forms, the root is the only chance node, and the chance actions are non-redundant in the following sense. There is an information set that follows a_0 ($\bar{a}_0$) but not a_0' ($\bar{a}_0'$). Consider the bijection $r : A_- \to \bar{A}_-$, $a \mapsto \bar{a}$ for $a \in \{L, R,$

[2]I wish to thank an anonymous referee for suggesting to look for such an example.

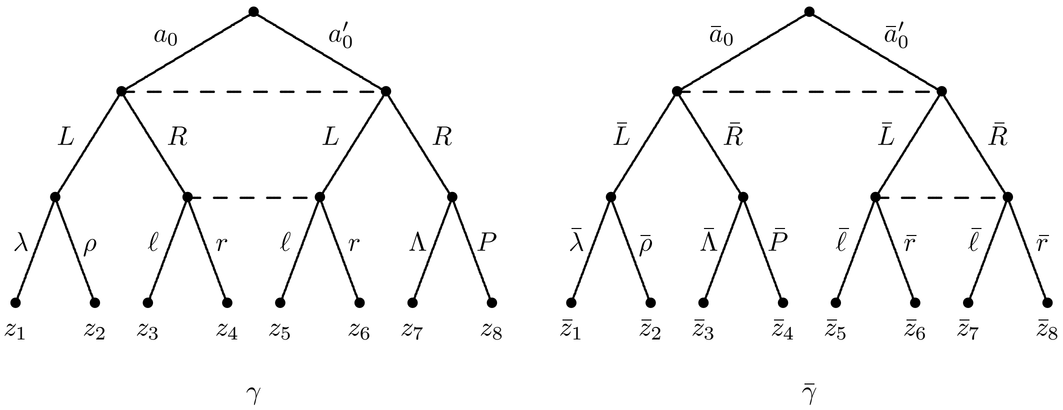

FIGURE 3.1. SWI forms that are not weakly isomorphic

$\ell, r, \Lambda, P, \lambda, \rho\}$. Obviously, this mapping satisfies **sISA** and **sPL**. In addition, it easy to check that r satisfies **sPTH** via the bijection $\theta : [Z] \to [\bar{Z}]$, $\theta([z_k]) = [\bar{z}_k]$ for $k = 1, 2, 5, 6$ and $\theta([z_3]) = [\bar{z}_7]$, $\theta([z_4]) = [\bar{z}_8]$, $\theta([z_7]) = [\bar{z}_3]$, $\theta([z_8]) = [\bar{z}_4]$. Hence, r is a SWI from γ to $\bar{\gamma}$. Yet, γ and $\bar{\gamma}$ cannot be WI: In γ, the action λ and the action Λ are contained in exactly one terminal history, and these terminal histories contain different chance actions, a_0 and a'_0, respectively. In contrast in $\bar{\gamma}$, just the actions $\bar{\lambda}$, $\bar{\rho}$, $\bar{\Lambda}$, and $\bar{P}$ are contained in exactly one terminal history where all these terminal histories contain the same chance action, $\bar{a}_0$.

3.4. SWI vs. ANF isomorphism. Obviously, any SWI $r : \Gamma \to \bar{\Gamma}$ induces an isomorphism $(\nu, (r|_{A_h})_{h \in H_-}) : \mathrm{ANF}(\Gamma) \to \mathrm{ANF}(\bar{\Gamma})$ where ν is determined via **sISA**. The converse, however, does not hold in general. Yet by Theorem 3.3, CA03 (Theorem 4.8) also applies to SWI: For $\mathcal{E}^* \cap \mathcal{E}^{\mathrm{nc}}$, SWI and ANF isomorphism are generically equivalent. [111]
Even though SWI largely disregards the chance mechanism, the following example reveals that this does not hold true for the whole set $\mathcal{E}^*$.

EXAMPLE 3.5. Consider $\gamma, \bar{\gamma} \in \mathcal{EF}$ in Figure 3.2 where just the roots are chance nodes (chance probabilities in brackets) and where the non-chance information sets are controlled by different players. γ and $\bar{\gamma}$ are not SWI: While in γ all RTH contain two actions, there is singleton one in $\bar{\gamma}$, $\bar{A}_-(\bar{z}_1) = \{\bar{L}\}$. Yet in the Appendix, we show that for all $\delta \in \mathbb{D}(\gamma)$ there is some $\bar{\delta} \in \mathbb{D}(\bar{\gamma})$ (and vice versa) such that $\gamma(\delta)$ and $\bar{\gamma}(\bar{\delta})$ are ANF isomorphic, contradicting genericity.

For SWI, CA03 (Theorem 4.8) can be extended to the set $\mathcal{E}^{\mathrm{reg}}$ ($\mathcal{EF}^{\mathrm{reg}}$) of regular games (forms). Let $H_-([z])$ denote the set of non-chance information sets corresponding to $A_-([z])$. A game (form) is called *regular* iff for all $[z], [z'] \in [Z]$, $[z] \neq [z']$, $H_-([z]) \cap H_-([z']) \neq \emptyset$ implies $\mathbf{A}([z]) \cap \mathbf{A}([z']) = \emptyset$, i.e., iff the RTH induced by the same strategy profile do not intersect. Of course, regularity is a strong property. Since $|[Z](\mathbf{a})| = 1$ in

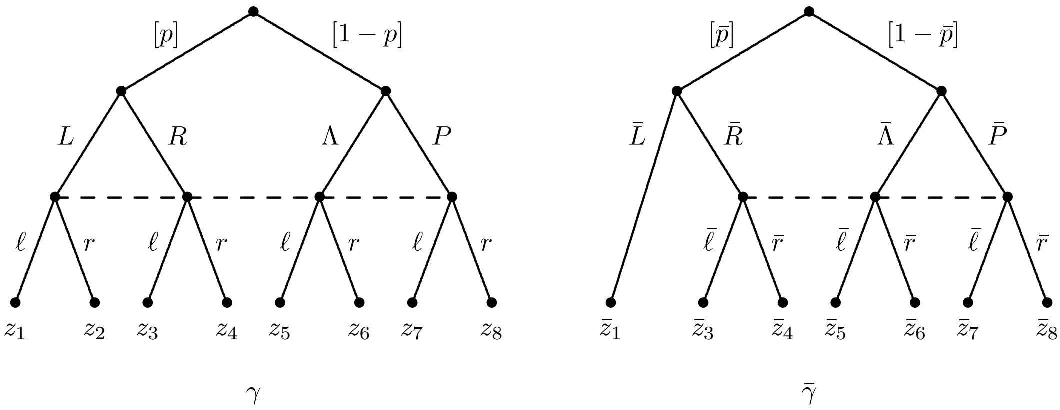

FIGURE 3.2. Non-SWI game forms

$\mathcal{E}^{\mathrm{nc}}$, we have $\mathcal{E}^{\mathrm{nc}} \subseteq \mathcal{E}^{\mathrm{reg}}$. For example, we obtain regular forms by connecting the root of two forms from $\mathcal{EF}^{\mathrm{nc}}$ with a chance node as the new root; the forms in Figure 3.2 are not regular. The proof of the following Theorem is referred to the Appendix.

THEOREM 3.6. *In $\mathcal{E}^{*} \cap \mathcal{E}^{reg}$, generically, any ANF isomorphism $\mathbf{f} = (\nu, (r_h)_{h \in H_-})$ from Γ to $\bar{\Gamma}$ induces a SWI $r : A_- \to \bar{A}_-$, $a \mapsto r_{V(a)}(a)$.*

3.5. Invariance under SWI. Since SWI preserves the (agent) normal form, the arguments for CA03 (Theorems 5.1 and 5.4) apply: SWI invariant perfect equilibria do always exist. Moreover, solution concepts that are based on the fixed (agent) normal form are SWI invariant, e.g. Nash and perfect equilibrium.

This argument does not work for sequential equilibrium because the Kreps & Wilson (1982, Proposition 6) characterization involves a sequence of payoff functions of the extensive game. Nevertheless, sequential equilibrium remains invariant of under SWI. But there are ANF isomorphic extensive games which are not SWI while any ANF isomorphism establishes a bijection of the set of sequential equilibria. By arguments in the proofs to Example 3.5 and of Theorem 3.7, one can show that the game forms in Figure 3.2 give rise to such games. A proof of the following Theorem can be found in the Appendix.

THEOREM 3.7. *Sequential equilibrium is SWI invariant.*

4. Concluding remarks [112]

In this note, we tried to answer the following question: Is it possible (via some concept of extensive game isomorphism) both to keep as less information as enables this concept and ANF isomorphism to be *generically* equivalent and to keep as much information as needed for the determination of *all* sequential equilibria?

Our answer is a partial one: For extensive games without chance mechanism, WI already does the job. SWI goes a little farther: Being equivalent to WI for games with out chance mechanism, it relaxes WI for general games in a way such that sequential equilibrium remains invariant. But even in spite of its disregard of the chance mechanism to a large extent and of the players' detailed preferences over individual terminal nodes, SWI makes only a small step towards generic equivalence which now extends to games that satisfy a strong regularity requirement. Even generically, the presence of a chance mechanism seems to enhance the structure of extensive games far beyond the ANF. Remains the question whether SWI can be further relaxed towards generic equivalence to ANF isomorphism without loosing the invariance of sequential equilibrium.

Acknowledgement

I thank research seminar participants at the Leipzig Graduate School of Mangement (HHL), at the University of Leipzig, and at the University of East Anglia for helpful discussions. Also, I am grateful to two anonymous referees whose critical comments improved the paper. Of course, the usual disclaimer applies.

I also wish to thank the School of Economics at the UEA for their hospitality during the final draft of the underlying paper. Financial support by the Deutsche Forschungsgemeinschaft is gratefully acknowledged. Last but not least, I am indebted to my parents, Peter and Gudrun Kotzan, for partially funding my fellowship at the UEA.

Appendix A

Proof of Lemma 3.2. (i) $[z] \in [Z](\mathbf{a}) \Leftrightarrow A_-([z]) \subseteq \{\mathbf{a}_h | h \in H_-\}$ by (2.1), $\Leftrightarrow r(A_-([z])) \subseteq \{r(\mathbf{a}_h) | h \in H_-\}$ by bijectivity of r, $\Leftrightarrow r(A_-([z])) \subseteq \left\{\mathbf{r}_{\nu(h)}(\mathbf{a}) | h \in H_-\right\}$ by **sPY**, $\Leftrightarrow \bar{A}_-(\theta([z])) \subseteq \left\{\mathbf{r}_{\nu(h)}(\mathbf{a}) | h \in H_-\right\}$ by **(3.1)**, $\Leftrightarrow \bar{A}_-(\theta([z])) \subseteq \{\mathbf{r}_{\bar{h}}(\mathbf{a}) | \bar{h} \in \bar{H}_-\}$ by bijectivity of r, $\Leftrightarrow \theta([z]) \in \left[\bar{Z}\right](\mathbf{r}(\mathbf{a}))$ by (2.1).

(ii) Let r be as in the Lemma. By $\mathbf{sPTH}^-$, r induces a bijection $\theta : [Z] \to \left[\bar{Z}\right]$. Consider $\mathbf{a} \in \mathbf{A}([z])$ and $a \in A_-([z])$. As $\Gamma \in \mathcal{E}^*$, there is some $a' \in A_{V(a)}$, $a' \neq a$. Consider $\mathbf{a}' \in \mathbf{A}$, $\mathbf{a}'_{V(a)} = a'$ and $\mathbf{a}'_h = \mathbf{a}_h$ for $h \neq V(a)$. Obviously, $[z] \notin [Z](\mathbf{a}')$. Suppose, $r(a) \notin \bar{A}_-(\theta([z]))$. We then had $\bar{A}_-(\theta([z])) \subseteq \left\{\mathbf{r}_{\bar{h}}(\mathbf{a}) | \bar{h} \in \bar{H}_-\right\} \setminus \{r(a)\}$ by **sPY**, $\mathbf{sPTH}^-$, and (2.1), $\subseteq \left\{\mathbf{r}_{\bar{h}}(\mathbf{a}') | \bar{h} \in \bar{H}_-\right\}$, i.e. $\theta([z]) \in \left[\bar{Z}\right](\mathbf{r}(\mathbf{a}'))$ by (2.1), contradicting $\mathrm{s}\mathbf{PTH}^-$. Hence, $r(A_-([z])) \subseteq \bar{A}_-(\theta([z]))$. Since the inverse r^{-1} satisfies **sISA** and $\mathbf{sPTH}^-$, the converse inclusion is immediate.

Proof to Example 3.5. For all assignments $\delta = (p, u)$, $p \in]0,1[$ and $u_i^k := u_i(z_k) \in \mathbb{R}$, $i \in \{1,2,3\}$, $k \in \{1,2,\ldots,8\}$ there is an assignment $\bar{\delta} = (\bar{p}, \bar{u})$, $\bar{p} \in]0,1[$ and $\bar{u}_i^k := \bar{u}_i(\bar{z}_k) \in \mathbb{R}$, $k \in \{1,3,\ldots,8\}$ (and vice versa) such that $r : A_- \to \bar{A}_-$, $a \mapsto \bar{a}$ (satisfying **sISA**) induces an isomorphism $\mathrm{ANF}(\gamma(\delta)) \to \mathrm{ANF}\left(\bar{\gamma}\left(\bar{\delta}\right)\right)$, i.e. satisfies **sPY**. Just set $p = \bar{p}$, $\bar{u}_i^3$

$= u_i^3 + \bar{u}_i^1 - u_i^1$, $\bar{u}_i^4 = u_i^4 + \bar{u}_i^1 - u_i^2$, $\bar{u}_i^5 = u_i^5 + \frac{p}{1-p}\left(u_i^1 - \bar{u}_i^1\right)$, $\bar{u}_i^6 = u_i^6 + \frac{p}{1-p}\left(u_i^2 - \bar{u}_i^1\right)$, $\bar{u}_i^7 = u_i^7 + \frac{p}{1-p}\left(u_i^1 - \bar{u}_i^1\right)$, $\bar{u}_i^8 = u_i^8 + \frac{p}{1-p}\left(u_i^2 - \bar{u}_i^1\right)$ or $u_i^k = \bar{u}_i^k$ for $k \neq 2$, and $u_i^1 = u_i^2$, respectively.

Proof of Theorem 3.6. We denote by $\text{prob}\,(z) := \prod_{a\in A_0\cap A(\psi(z))} p_{V(a)}(a)$ the probability [113]
that $z \in Z(\mathbf{a})$ is reached by $\mathbf{a}$ which gives

(5.1) $$u_i(\mathbf{a}) = \sum_{z\in Z(\mathbf{a})} \text{prob}\,(z)\, u_i(z) = \sum_{[z]\in[Z](\mathbf{a})} v_i([z]) \qquad i \in I_-, \mathbf{a} \in \mathbf{A}$$

where $v_i([z]) := \sum_{z'\in[z]} \text{prob}\,(z')\, u_i(z')$ is called player i's *valuation* of $[z]$. Since we wish to prove a generic result within $\mathcal{E}^*$, we are allowed to focus on assignments with the following properties: (*) For all $i \in I_-$ and $\chi : [Z] \to \{0, \pm1, \pm2\}$, $\sum_{[z]\in[Z]} \chi([z])\, v_i([z]) = 0$ implies $\chi([z]) = 0$ for all $[z] \in [Z]$. (**) The players' preferences are pairwise different, i.e. there is no positive affine transformation between the payoff functions of any two players.

Let $\mathbf{f} = (\nu, (r_h)_{h\in H_-})$ be an isomorphism from ANF (Γ) to ANF $(\bar{\Gamma})$. The bijection $r : A_- \to \bar{A}_-$, $a \mapsto r_{V(a)}(a)$ then satisfies **sISA** and **sPY**. By (**) and **sPY,** r induces the bijection $\pi : I_- \to \bar{I}_-$, $i(h) \mapsto \bar{\imath}(\nu(h))$ which satisfies **sPL**.

Remains show that there is a bijection $\theta : [Z] \to [\bar{Z}]$ that satisfies (3.1), hence **sPTH**. Consider the correspondences $Y : [Z] \rightrightarrows [\bar{Z}]$ and $\bar{Y} : [\bar{Z}] \rightrightarrows [Z]$,

(5.2a) $$Y([z]) := \{[\bar{z}] \in [\bar{Z}] \,|\, \bar{A}_-([\bar{z}]) \subseteq r(A_-([z]))\}$$

(5.2b) $$\bar{Y}([\bar{z}]) := \{[z] \in [Z] \,|\, r(A_-([z])) \subseteq \bar{A}_-([\bar{z}])\}$$

By (5.2), $[\bar{z}] \in Y([z])$ and $[z'] \in \bar{Y}([\bar{z}])$ imply $r(A_-([z'])) \subseteq \bar{A}_-([\bar{z}]) \subseteq r(A_-([z]))$, hence $A_-([z']) \subseteq A_-([z])$. Regularity then implies $[z'] = [z]$, hence $r(A_-([z])) = \bar{A}_-([\bar{z}])$. I.e., if both Y and $\bar{Y}$ are nowhere empty then both are single-valued and inverse to each other. Thus, $\{\theta([z])\} = Y([z])$ determines the desired bijection θ. In view of the bijectivity of r, Y and $\bar{Y}$ are defined symmetrically. Therefore, it suffices to show $Y([z]) \neq \emptyset$ for all $[z] \in [Z]$. For $H_-([z]) = H_-$, we have $\mathbf{A}([z]) = \{\mathbf{a}\}$ and $[\bar{Z}](\mathbf{r}(\mathbf{a})) \subseteq Y([z])$. For $H_-([z]) \subsetneq H_-$, we proceed by a series of claims where the first one merely is a restatement of (2.1) and last one implies $Y([z]) \neq \emptyset$.

Claim 1: $[z] \in [Z](\mathbf{a})$ iff $\mathbf{a}_h \in A_-([z])$ for all $h \in H_-([z])$.

Claim 2: $[Z](\mathbf{a}') \subseteq [Z](\mathbf{a})$ implies $[Z](\mathbf{a}') = [Z](\mathbf{a})$.

It suffices to show that $Z(\mathbf{a}') \subseteq Z(\mathbf{a})$ implies $Z(\mathbf{a}') = Z(\mathbf{a})$. For $z^* \in Z(\mathbf{a})$, by (2.1), there is some $\mathbf{a}_0^* \in \mathbf{A}_0$ such that $z^* = z(\mathbf{a}, \mathbf{a}_0^*)$. We then have $z(\mathbf{a}', \mathbf{a}_0^*) \in Z(\mathbf{a}') \subseteq Z(\mathbf{a})$, i.e. by (2.1), there is some $\mathbf{a}_0 \in \mathbf{A}_0$ such that $z(\mathbf{a}', \mathbf{a}_0^*) = z(\mathbf{a}, \mathbf{a}_0)$. By CA03 (Equation (2.3)), we then have $z(\mathbf{a}', \mathbf{a}_0^*) = z(\mathbf{a}, \mathbf{a}_0^*)$ and therefore $z^* \in Z(\mathbf{a}')$.

Claim 3: If (a) $H_-([z]) \cap H_-([z']) = \emptyset$ and (b) $H_-([z]) \cap H_-([z'']) \neq \emptyset$ then (c) $H_-([z']) \cap H_-([z'']) = \emptyset$.

Suppose on the contrary that $[z], [z'], [z''] \in [Z]$ satisfy (a) and (b) but not (c). Then there are $h \in H_-([z])$ and $h' \in H_-([z'])$ that intersect $\psi(z'')$ as close as possible to the root, respectively. Set $\{x\} := h \cap \psi(z'')$ and $\{x'\} := h' \cap \psi(z'')$. W.l.o.g. we assume $x' \lhd x$. By the choice of h, there are $\mathbf{a}^{\#} \in \mathbf{A}([z])$ and $[z^{\#}] \in [Z](\mathbf{a}^{\#})$ such that $x \in \psi(z^{\#})$. We then have $[z], [z^{\#}] \in [Z](\mathbf{a}^{\#})$ and $h \in H_-([z]) \cap H_-([z^{\#}])$. By $x' \lhd x$, we also have $h' \in H_-([z^{\#}])$, and by (a), $h' \notin H_-([z])$, hence $[z] \neq [z^{\#}]$, contradicting regularity.

Fix some $[z]$ and $\mathbf{a} \in \mathbf{A}([z])$. Since $\Gamma \in \mathcal{E}^*$, there is some $\mathbf{a}^\bullet \in \mathbf{A}$ such that $\mathbf{a}^\bullet_h \neq \mathbf{a}_h$, [114]
$h \in H_-$. Setting

$$(5.3) \qquad H^*_-([z]) := \bigcup_{[z'] \in [Z](\mathbf{a}^\bullet) : H_-([z']) \cap H_-([z]) \neq \emptyset} H_-([z']),$$

we construct $\mathbf{a}^\circ, \mathbf{a}^* \in \mathbf{A}$ as follows:

$$(5.4) \qquad \mathbf{a}^\circ_h = \begin{cases} \mathbf{a}_h & , h \in H_-([z]) \\ \mathbf{a}^\bullet_h & , h \in H_- \backslash H_-([z]) \end{cases} \qquad \mathbf{a}^*_h = \begin{cases} \mathbf{a}^\bullet_h & , h \in H^*_-([z]) \\ \mathbf{a}_h & , h \in H_- \backslash H^*_-([z]) \end{cases}$$

Claim 4: $H^*_-([z]) \neq \emptyset$.

Suppose on the contrary, $H^*_-([z]) = \emptyset$, i.e. by (5.3) there is no $[z'] \in [Z](\mathbf{a}^\bullet)$ such that $H_-([z']) \cap H_-([z]) \neq \emptyset$. Then $[Z](\mathbf{a}^\bullet) \subseteq [Z](\mathbf{a}^\circ)$ by (5.4) and *Claim 1*, hence $[Z](\mathbf{a}^\bullet) = [Z](\mathbf{a}^\circ)$ by *Claim 2*. By (5.4) and *Claim 1*, however, $[z] \in [Z](\mathbf{a}^\circ)$ but $[z] \notin [Z](\mathbf{a}^\bullet)$. A contradiction.

Claim 5: For all $i \in I_-$, $u_i(\mathbf{a}) - u_i(\mathbf{a}^\circ) - u_i(\mathbf{a}^*) + u_i(\mathbf{a}^\bullet) = 0$.

Set $M_1 := \{[z]\}$, $M_2 := [Z](\mathbf{a}) \setminus \{[z]\}$, $M_3 := \{[z'] \in [Z](\mathbf{a}^\bullet) \,|\, H_-([z']) \subseteq H^*_-([z])\}$, and $M_4 := \{[z'] \in [Z](\mathbf{a}^\bullet) \,|\, H_-([z']) \cap H^*_-([z]) = \emptyset\}$. In the following, we show (i) $[Z](\mathbf{a}) = M_1 \cup M_2$, (ii) $[Z](\mathbf{a}^\circ) = M_1 \cup M_4$, (iii) $[Z](\mathbf{a}^*) = M_3 \cup M_2$, and (iv) $[Z](\mathbf{a}^\bullet) = M_3 \cup M_4$. By (5.1), this proves the claim.

By $[z] \in [Z](\mathbf{a})$, (i) is immediate. By (5.3), either $H_-([z']) \subseteq H^*_-([z])$ or $H_-([z']) \cap H^*_-([z]) = \emptyset$ for $[z'] \in [Z](\mathbf{a}^\bullet)$. This proves (iv). By (5.4) and *Claim 1*, we have $M_1 \subseteq [Z](\mathbf{a}^\circ)$. If $[z'] \in [Z](\mathbf{a}^\circ) \backslash M_1$ then $H_-([z']) \cap H_-([z]) = \emptyset$ by regularity. Then (5.4), (5.3), and *Claim 1* imply $[z'] \in M_4$. This proves (ii). By (5.4), (5.3), and *Claim 1*, we have $M_3 \subseteq [Z](\mathbf{a}^*)$. Together with regularity, we have $H_-([z']) \subseteq H_- \backslash H^*_-([z])$ for $[z'] \in [Z](\mathbf{a}^*) \backslash M_3$, hence $[z'] \in [Z](\mathbf{a}) = M_1 \cup M_2$. *Claim 4* and regularity imply $[z'] \in M_2$, i.e. $[Z](\mathbf{a}^*) \backslash M_3 \subseteq M_2$. If $[z'] \in M_2$ and $[z''] \in M_3$ then $H_-([z']) \cap H_-([z]) = \emptyset$ by regularity, and $H_-([z'']) \cap H_-([z]) \neq \emptyset$ by definition of M_3. *Claim 3* then implies $H_-([z']) \cap H_-([z''])$ $= \emptyset$. Then, again by (5.4), (5.3), and *Claim 1*, we have $[z'] \in [Z](\mathbf{a}^*) \backslash M_3$, hence $M_2 \subseteq [Z](\mathbf{a}^*) \backslash M_3$ which proves (iii).

Claim 6: $[\bar{Z}](\mathbf{r}(\mathbf{a})) \cap [\bar{Z}](\mathbf{r}(\mathbf{a}^\circ)) \neq \emptyset$ where $\mathbf{r}$ is induced by r via **sPY**.

By (5.2), (5.4), and *Claim 1*, we have $Y([z]) = [\bar{Z}](\mathbf{r}(\mathbf{a})) \cap [\bar{Z}](\mathbf{r}(\mathbf{a}^{\circ}))$. Hence, the claim shows $Y([z]) \neq \emptyset$. Suppose on the contrary, $[\bar{Z}](\mathbf{r}(\mathbf{a})) \cap [\bar{Z}](\mathbf{r}(\mathbf{a}^{\circ})) = \emptyset$. Consider any $[\bar{z}] \in [\bar{Z}](\mathbf{r}(\mathbf{a}))$, hence $[\bar{z}] \notin [\bar{Z}](\mathbf{r}(\mathbf{a}^{\circ}))$.

Suppose there is some $\bar{h}' \in \bar{H}_{-}([\bar{z}])$ such that $\bar{h}' \in \nu\left(H_{-}^{*}([z])\right)$. Then by (5.4) and **sPY**, $\mathbf{r}_{\bar{h}'}(\mathbf{a}) \neq \mathbf{r}_{\bar{h}'}(\mathbf{a}^{*}) = \mathbf{r}_{\bar{h}'}(\mathbf{a}^{\bullet})$, hence by *Claim 1*, $[\bar{z}] \notin [\bar{Z}](\mathbf{r}(\mathbf{a}^{*})), [\bar{Z}](\mathbf{r}(\mathbf{a}^{\bullet}))$. Since $\mathbf{r}$ satisfies **sPY**, for all $i \in I_{-}$ and $\mathbf{a}' \in \mathbf{A}$ there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that $\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}'))$ $= \alpha_i u_i(\mathbf{a}') + \beta_i$. Hence by *Claim 5*,

$$\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) - \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^{\circ})) - \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^{*})) + \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^{\bullet})) = 0. \tag{5.5}$$

Express (5.5) by valuations according to (5.1). Since $[\bar{z}]$ is contained in $[\bar{Z}](\mathbf{r}(\mathbf{a}))$ only, the coefficient of $\bar{v}_{\bar{\imath}(\nu(h))}([\bar{z}])$ is 1 while all other coefficients are between -2 and 2, contradicting (*), i.e. genericity.

Remains the possibility that $\bar{H}_{-}(\bar{z}) \subseteq \bar{H}_{-} \backslash \nu\left(H_{-}^{*}([z])\right)$. Then by (5.4), *Claim 1*, and **sPY**, $[\bar{z}] \in [\bar{Z}](\mathbf{r}(\mathbf{a}^{*}))$, hence $[\bar{Z}](\mathbf{r}(\mathbf{a})) \subseteq [\bar{Z}](\mathbf{r}(\mathbf{a}^{*}))$ (since $[\bar{z}]$ was arbitrary) and therefore $[\bar{Z}](\mathbf{r}(\mathbf{a})) = [\bar{Z}](\mathbf{r}(\mathbf{a}^{*}))$ by *Claim 2*. By *Claims 4* and *5* ((i), (iii)), and regularity, however, $[Z](\mathbf{a}) \neq [Z](\mathbf{a}^{*})$. Arguments similar to those for the other case show [115]
that this contradicts genericity.

Proof of Theorem 3.7. We denote by μ^{*} the mapping that assigns to $b' \in B^{0}$ the system of beliefs $\mu^{*}(b')$ associated with b' according to Bayes' rule. Let (μ, b) be a sequential equilibrium of $\Gamma \in \mathcal{E}$. By Kreps & Wilson (1982, Proposition 6), there is a sequence $\left(b^{k}, u^{k}\right)$, $b^{k} \in B^{0}$, $u^{k} \in \mathbb{R}^{I_{-} \times Z}$ such that $b = \lim_{k\to\infty} b^{k}$, $\mu = \lim_{k\to\infty} \mu^{*}\left(b^{k}\right)$, $u = \lim_{k\to\infty} u^{k}$ and $u_i^{k}\left(b_i b_{-i}^{k}\right) \geq u_i^{k}\left(b_i' b_{-i}^{k}\right)$ for all $k \in \mathbb{N}$, $i \in I_{-}$, and $b_i' \in B$.

Further, let r be a SWI from Γ to $\bar{\Gamma} \in \mathcal{E}$ which induces bijections $\pi : I_{-} \to \bar{I}_{-}$, $\nu : H_{-} \to \bar{H}_{-}$, $\theta : [Z] \to [\bar{Z}]$, $\mathbf{r} : \mathbf{A} \to \bar{\mathbf{A}}$ such that for all $i \in I_{-}$ there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that $\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i(\mathbf{a}) + \beta_i$ for all $\mathbf{a} \in \mathbf{A}$. Since $\bar{\mu}^{*}$ is continuous, there is some system of beliefs $\bar{\mu}$ of $\bar{\Gamma}$ such that $\lim_{k\to\infty} \bar{\mu}^{*}\left(\mathbf{r}\left(b^{k}\right)\right) = \bar{\mu}$. We show that $(\bar{\mu}, \mathbf{r}(b))$ is a sequential equilibrium.

Fix any payoff function $\upsilon \in \mathbb{R}^{Z}$ and consider the following system of linear equations where the payoff function $\bar{\upsilon} \in \mathbb{R}^{\bar{Z}}$ is variable:

$$\bar{\upsilon}(\mathbf{r}(\mathbf{a})) = \sum_{\bar{z} \in \bar{Z}(\mathbf{r}(\mathbf{a}))} \operatorname{prob}(\bar{z})\, \bar{\upsilon}(\bar{z}) = \sum_{z \in Z(\mathbf{a})} \operatorname{prob}(z)\, \upsilon(z) = \upsilon(\mathbf{a}) \qquad \mathbf{a} \in \mathbf{A} \tag{5.6}$$

Let $\bar{\Upsilon}$ denote the correspondence $\mathbb{R}^{Z} \rightrightarrows \mathbb{R}^{\bar{Z}}$ which assigns to υ the set $\bar{\Upsilon}(\upsilon)$ of solutions of (5.6). Using Lemma 3.2 (i), one shows that $\bar{\upsilon}(\upsilon) \in \mathbb{R}^{\bar{Z}}$,

$$\bar{\upsilon}(\upsilon)(\bar{z}) := \frac{\sum_{z \in \theta^{-1}([\bar{z}])} \operatorname{prob}(z)\, \upsilon(z)}{\sum_{\bar{z}' \in [\bar{z}]} \operatorname{prob}(\bar{z}')} \qquad , \bar{z} \in \bar{Z}$$

satisfies (5.6). Hence, $\bar{\Upsilon}(v)$ is non-empty for all $v \in \mathbb{R}^{Z}$. Moreover, the set $\bar{\Upsilon}(v)$ is an affine subspace $\bar{v}^* + \bar{\Upsilon}_0 \subseteq \mathbb{R}^{\bar{Z}}$ where $\bar{v}^* \in \bar{\Upsilon}(v)$ and $\bar{\Upsilon}_0$ denotes the solution set of the homogenous system associated with (5.6). Since the right side of (5.6) is continuous in v, $\bar{\Upsilon}$ is continuous.

By assumption, we have $\bar{u}_{\pi(i)} \in \bar{\Upsilon}(\alpha_i u_i + \beta_i)$ for all $i \in I_-$. Since $\lim_{k\to\infty} u_i^k = u_i$ and $\bar{\Upsilon}$ is continuous, there is a sequence $(\bar{u}^k_{\pi(i)})_{k\in\mathbb{N}}$, $\bar{u}^k_{\pi(i)} \in \bar{\Upsilon}\left(\alpha_i u_i^k + \beta_i\right)$ such that $\lim_{k\to\infty} \bar{u}^k_{\pi(i)} = \bar{u}_{\pi(i)}$. By (5.6) and (5.1), we then have $\bar{u}^k_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i^k(\mathbf{a}) + \beta_i$ for all $\mathbf{a} \in \mathbf{A}$, $i \in I_-$, and $k \in \mathbb{N}$, hence

$$\bar{u}^k_{\pi(i)}(\mathbf{r}(b)) = \alpha_i u_i^k(b) + \beta_i, \qquad b \in B. \tag{5.7}$$

Since $\mathbf{r}$ is continuous, $\lim_{k\to\infty} \mathbf{r}\left(b^k\right) = \mathbf{r}(b)$. Suppose there were some $k \in \mathbb{N}$, $\bar{\imath} \in \bar{I}_-$, $\bar{b}'_{\bar{\imath}} \in \bar{B}_{\bar{\imath}}$ such that

$$\bar{u}^k_{\bar{\imath}}\left(\mathbf{r}\left(b_{\pi^{-1}(\bar{\imath})} b^k_{-\pi^{-1}(\bar{\imath})}\right)\right) < \bar{u}^k_{\bar{\imath}}\left(\bar{b}'_{\bar{\imath}} \mathbf{r}_{-\bar{\imath}}\left(b^k\right)\right)$$

where $\bar{b}'_{\bar{\imath}} \mathbf{r}_{-\bar{\imath}}\left(b^k\right)$ denotes the behavior strategy profile where all players follow $\mathbf{r}\left(b^k\right)$ except for $\bar{\imath}$ who follows $\bar{b}'_{\bar{\imath}}$, analogously for $b_{\pi^{-1}(\bar{\imath})} b^k_{-\pi^{-1}(\bar{\imath})}$. By (5.7) we then had

$$u^k_{\pi^{-1}(\bar{\imath})}\left(b_{\pi^{-1}(\bar{\imath})} b^k_{-\pi^{-1}(\bar{\imath})}\right) < u^k_{\pi^{-1}(\bar{\imath})}\left(\mathbf{r}^{-1}_{\pi^{-1}(\bar{\imath})}\left(\bar{b}'_{\bar{\imath}}\right) b^k_{-\pi^{-1}(\bar{\imath})}\right)$$

with the interpretation of the arguments as above. Since this contradicts the assumptions [116] on $\left(b^k, u^k\right)$, the sequence $\left(\mathbf{r}\left(b^k\right), \bar{u}^k\right)$ establishes $(\bar{\mu}, \mathbf{r}(b))$ to be a sequential equilibrium. Since the inverse r^{-1} also is a SWI, this proves the claim.

References

Casajus, A. (2003). Weak isomorphism of extensive games, *Mathematical Social Sciences* **46**(3): 267–290.

Casajus, A. (2005). Super weak isomorphism of extensive games, *working paper, revised version of #04/AC/01*, Universität Leipzig, Germany. http://www.uni-leipzig.de/˜micro/swieg.pdf.

Elmes, S. & Reny, P. J. (1994). On the strategic equivalence of extensive form games, *Journal of Economic Theory* **62**(1): 1–23.

Govindan, S. & Wilson, R. (2004). Axiomatic justification of stable equilibria, University of Iowa and Stanford University.

Harsanyi, J. C. & Selten, R. (1988). *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge, MA.

Kohlberg, E. & Mertens, J.-F. (1986). On the strategic stability of equilibria, *Econometrica* **54**(5): 1003–1037.

Kreps, D. M. & Wilson, R. (1982). Sequential equilibria, *Econometrica* **50**(4): 863–94.

Myerson, R. B. (1978). Refinements of the Nash equilibrium concept, *International Journal of Game Theory* **7**(2): 73–80.

Peleg, B., Rosenmüller, J. & Sudhölter, P. (1999). The canonical extensive form of a game form: Symmetries, *in* A. Alkan, C. Aliprantis & N. Yannelis (eds), *Current Trends in Economics: Theory and Applications*, Springer, pp. 367–387.

Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory* **4**(1): 25–55.