The Wiese Value and the Gloves Game: A Technical Note

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André Casajus[1]

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Abstract

In this note, we show that under replication the Wiese (2003) payoffs for the gloves game with balanced coalition structures converge to the core

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[1]Universität Leipzig, Wirtschaftswissenschaftliche Fakultät, Professur für Mikroökonomik, PF 100920, D-04009 Leipzig, Germany. Email: casajus@wifa.uni-leipzig.de

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1. Introduction

Wiese (2003) suggests a component efficient CS-value which captures the outside options of the players whithin a given coalition structure. For the gloves game with a balanced coalition structure it has been conjectured that under replication the Wiese payoffs converge to the core (Rosenmüller, privat communication). In this note, we show that this is immediate from the Shapley and Shubik (1969) convergence result for the Shapley payoffs.

2. Basic definitions

A (TU) game is a pair (N, v) consisting of a non-empty and finite set of players N, $n = |N|$ and the coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. For $i \in N$ and $K \subset N$, $v(\{i\})$ and $v(K)$ are called the worth of i or K, respectively. A value is an operator φ that assigns payoff vectors to all games, $\varphi : (N, v) \mapsto \mathbb{R}^N$. A coalition structure for (N, v) is a partition $\mathcal{P} \subset 2^N$ where $\mathcal{P}(i)$ denotes the cell containing player i. In general, subsets of N are called coalitions; elements of $\mathcal{P}$ are referred to as structural coalitions. A CS-game is a game together with a coalition structure, $(N, v, \mathcal{P})$. A CS-value is an operator φ that assigns payoff vectors to all CS-games, $\varphi : (N, v, \mathcal{P}) \mapsto \mathbb{R}^N$. For $K \subset N$, we denote by $\varphi_K(N, v, \mathcal{P})$ the sum $\sum_{i \in K} \varphi_i(N, v, \mathcal{P})$. When it is clear which game is meant, we sometimes drop the argument of the value operator.

An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the obvious interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players not after i in σ is denoted $K_i(\sigma) = \{j : \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i in σ is defined as $MC_i(\sigma) = v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley value φ^{Sh} is defined as the average marginal contribution over all orderings of players, $\varphi_i^{\text{Sh}}(N, v) = |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i(\sigma)$.

3. The Wiese value

Originally, the Wiese (2003) value is defined in terms of marginal contributions:

$$W_i(N, v, \mathcal{P}) = \frac{1}{|\Sigma(N)|} \sum_{\sigma \in \Sigma(N)} \begin{cases} v(\mathcal{P}(i)) - \sum\limits_{j \in \mathcal{P}(i) \setminus \{i\}} MC_j(\sigma) & , \mathcal{P}(i) \subset K_i(\sigma) \\ MC_i(\sigma) & , \mathcal{P}(i) \not\subset K_i(\sigma) \end{cases}$$

Applying the Shapley-value formula, we obtain the following equation

(3.1)

$$W_i(N,v,\mathcal{P}) = \varphi_i^{\mathrm{Sh}}(N,v) + \frac{v(\mathcal{P}(i))}{|\mathcal{P}(i)|} - \frac{1}{|\mathcal{P}(i)|}\frac{1}{|\Sigma_i(N,\mathcal{P})|}\sum_{\sigma\in\Sigma_i(N,\mathcal{P})}\sum_{j\in\mathcal{P}(i)} MC_j(\sigma)$$

where $\Sigma_i(N,\mathcal{P}) \subset \Sigma(N)$ such that $\sigma \in \Sigma_i(N,\mathcal{P})$ iff $\sigma(i) \geq \sigma(j)$ for all $j \in \mathcal{P}(i)$. Note that $|\Sigma(N)| = |\mathcal{P}(i)|\,|\Sigma_i(N,\mathcal{P})|$.

4. The gloves game

Shapley and Shubik (1969) consider as simple market game—the gloves game $[\lambda,\rho]$. There are $\lambda > 0$ left-glove holders (ℓ) in L and $\rho > 0$ right-glove holders (r) in R. The coalition function is given by $v(K) = \min(|R\cap K|, |L\cap K|)$ for $K \subset R \dot\cup L =: N$, i.e. the worth of a coalition is the number of its matching pairs of gloves. For symmetry reasons, we focus on the case $\rho \geq \lambda$. The Shapley payoffs are given by

$$\varphi_r^{\mathrm{Sh}}(\lambda,\rho) = \frac{1}{2} - \frac{\rho-\lambda}{2\rho}\sum_{k=0}^{\lambda}\frac{\binom{\lambda}{k}}{\binom{\rho+k}{k}}, \qquad \varphi_\ell^{\mathrm{Sh}}(\lambda,\rho) = \frac{1}{2} + \frac{\rho-\lambda}{2\lambda}\sum_{k=1}^{\lambda}\frac{\binom{\lambda}{k}}{\binom{\rho+k}{k}}$$

Shapley and Shubik (1969) show that under replication ($[\alpha\lambda,\alpha\rho], \alpha \to \infty$) the Shapley value converges to the core. If $\rho > \lambda$, the Shapley payoff of a left-glove holder converges to 1 and that one of right-glove holder to 0. For $\rho = \lambda$, both payoffs are $\frac{1}{2}$. We consider balanced coalition structures $\mathcal{P}$, i.e. coalition structures where all structural coalitions containing left-glove holders contain the same number of right-glove holders. For these structures, the core obviously is component efficient. Hence in the limit, the Shapley payoffs become component efficient. Note that is not the case for unbalanced structural coalitions containing left-glove holders. Whereas the worth "counts" the matching pairs, at the limit, the Shapley value "counts" the number of left-glove holders within a structural coalition.

5. Convergence

Theorem 5.1. *For balanced coalition structures and $[\alpha\lambda,\alpha\rho]$, $\alpha\to\infty$, the Wiese payoffs converge to the core.*

Proof. For $\rho = \lambda$, the claim is immediate from a symmetry argument. Let now be $\rho > \lambda$. Since the marginal contribution in the gloves game either is 0 or 1, the Shapley payoff is the probability that this marginal contribution is 1 if we consider the set of all orders for $[\alpha\lambda, \alpha\rho]$. Since the Shapley payoffs converge to the core, this probability converges to 1 for left-glove holders and to 0 for right-glove holders. Consider some balanced structural coalition $\mathcal{P}(i)$. The probability that i is the last player from $\mathcal{P}(i)$ is $|\mathcal{P}(i)|^{-1}$ and fixed under the replication. Therefore, the probability that the marginal contribution is 1 conditional on that i is the last player from $\mathcal{P}(i)$, again, is 1 for a left-glove holder and is 0 for a right-glove holder. Hence, we have

$$\lim_{\alpha\to\infty}\left(\frac{1}{|\Sigma([\alpha\lambda,\alpha\rho])|}\sum_{\substack{\sigma\in\Sigma([\alpha\lambda,\alpha\rho])\\ \mathcal{P}(i)\subset K_i(\sigma)}} MC_\ell(\sigma)\right) = \frac{1}{|\mathcal{P}(i)|}$$

and

$$\lim_{\alpha\to\infty}\left(\frac{1}{|\Sigma([\alpha\lambda,\alpha\rho])|}\sum_{\substack{\sigma\in\Sigma([\alpha\lambda,\alpha\rho])\\ \mathcal{P}(i)\subset K_i(\sigma)}} MC_r(\sigma)\right) = 0.$$

By $\rho > \lambda$ and by construction, we have $v(\mathcal{P}(i)) = |\mathcal{P}(i) \cap L|$. In view of (3.1), the Wiese payoffs converge to the Shapley payoffs, hence to the core. ∎

References

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